Some other models of consensus

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October 2009

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Other sources of consensus-like phenomena:

1. ‘Population algorithms’

2. Learning in games

3. Coordination of autonomous agents

Typically involve stochastic dynamics, analyzed by the ‘o.d.e. limit’.

Example 1: Ant Colony Optimization

Intuition:

- Ants sample alternative paths to food source and deposit pheromone when they do so.
- Future ants prefer high pheromone paths.
- This builds up a reinforcement mechanism that concentrates ant population on the optimal path.

$p_i(t) :=$ probability of choosing path i at time t

$f_i(p) :=$ stationary expectation of the pheromone level on path i if $p(t) \equiv p$

Assume: $p_i > p_j,\ j \neq i \Longrightarrow f_i(p) > f_j(p)\ \forall\ j \neq i.$

Probability increased / decreased in proportion to the excess / shortfall of average pheromone level against the population average.

Thus:

$$\dot{p}_i(t) = p_i(t)[f_i(p(t)) - \sum_j p_j(t) f_j(p_j(t))].$$

'*Replicator dynamics*' ($p_i(t) \approx$ population share of species i). Can show: If $p_i(0) > p_j(0)\ \forall\ j \neq i$, then $p_i(t) \uparrow 1$.

$\Longrightarrow$ ants concentrate on a single path.

Fact: Initial bias favors the shortest path with a probability that decreases to zero as the number of ants $\uparrow \infty$.

Example 2: Learning in repeated games:

- symmetric payoff matrix $A := [[a_{ij}]]$

- $p_i(t) :=$ the fraction of population using action i

- change from action i to j with probability

$$q_{ij}(p(t)) := exp(- \beta(\sum_k a_{ik} p_k(t) - \sum_k a_{jk} p_k(t))^+).$$

The '*master equation*' is

$$\dot{p}_i(t) = \sum_{j \neq i} q_{ji}(p(t))p_j(t) - p_i(t) \sum_{j \neq i} q_{ij}(p(t))$$

$H(p) := -\frac{1}{2}\sum_{i,j} a_{ij}p_ip_j + \beta^{-1}\sum_i p_i \log(p_i)$ serves as a Liapunov function $\Longrightarrow p(t) \to$ a local minimum of H.

p^* a local minimum $\Longleftrightarrow p_i^* = Z^{-1} exp(\beta \sum_k a_{ik}p_k^*) \; \forall \; i$.

As $\beta \uparrow \infty$, $p^* \to$ 'Nash' $\widehat{p}$:

$$\widehat{p}^T A \widehat{p} \geq q^T A \widehat{p} \; \forall \; q \neq \widehat{p}$$

Example 3: A coordination problem:

Consider solving the scalar equation $f(x) = 0$ by the 'algorithm' $\dot{x}(t) = f(x(t))$.

Suppose $x(t) \to H := \{x : f(x) = 0\}$, assumed to be finite.

'Distributed' version: $\dot{x}_i(t) = f(x_i(t)), 1 \leq i \leq N$.

Want: $x_i(t)$ converge to the *same* element of H.

Consider

$$\dot{x}(t) = h(x(t)) := (P - I)x(t) + F(x(t)), \qquad (1)$$

where $P := [[p_{ij}]]$ is an irreducible stochastic matrix and $F([x_1, \cdots, x_N]^T) := [f(x_1(t)), \cdots, f(x_N(t))]^T$.

Assume bounded trajectories. $\frac{\partial h_i}{\partial x_j} \geq 0$ and the positive terms form an irreducible matrix.

$\Longrightarrow$ 'cooperative system' $\Longrightarrow$ converges generically to the set $G := \{x : h(x) = 0\}$ (Hirsch).

If $x^* \in H$, $[x^*, , x^*, \cdots, x^*]^T \in G$.

If $x_i \in H, x_i \neq x_j$ for some $i \neq j$, then $x = [x_1, \cdots, x_N]^T \notin G$.

It is possible that $x = [x_1, \cdots, x_N]^T \in G$ with x_i's not necessarily in H. If the system

$$\sum_j p_{ij} z_j - z_i + f(z_i) = 0 \ \forall \ i,$$

has no non-constant solution, then this does not occur.

Other examples:

- genetic algorithms, particle swarm algorithms, . . .
- technology adoption, herd behavior, evolution of conventions, . . .
- coordination and control of robots, UAVs, mobile sensors, . . .

Forces leading to consensus:

1. explicit reinforcement and / or penalty mechanism

2. ‘cooperative’ features such as averaging

3. symmetry of the problem.