Discussion of
Steve Shreve's Markov Lecture on

# "Mixing Models to Capture Stock Price Volatility"

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## First-Order Equivalent Queues

**E8 Exercise.** Let $Q_t$ be a simple queue with $\mathscr{F}_t$-parameters $\lambda_t$ and $\mu_t$ where $\lambda_t$ and $\mu_t$ are bounded. Define $p_n(t)$, $\lambda_n(t)$, and $\mu_n(t)$ by …

*Comment*. The queue $Q_t$ is said to be *first-order equivalent* to a (nonhomogeneous) markovian queue $\tilde{Q}_t$ with parameters $\lambda_{\tilde{Q}_t}(t)$ and $\mu_{\tilde{Q}_t}(t)$: indeed, if $Q_t$ and $\tilde{Q}_t$ have the same initial distribution, then they have the same distribution at any time $t \geq 0$. In particular, if $Q_t$ is in equilibrium, i.e. if $p_n(t) \equiv p_n$, $t \geq 0$, $n \geq 0$, then it is equivalent to a birth-and-death process with parameters $\lambda_n$ and $\mu_n$ determined by

$$\lambda_n p_n = \mu_{n+1} p_{n+1}, \tag{3.10}$$

and moreover, if $p_n > 0$ for all $n \geq 0$, then $\lambda_n$ can be chosen to be independent of $n$ (Poissonian input).

# First-Order Equivalent Queues

**E8 Exercise.** Let $Q_t$ be a simple queue with $\mathscr{F}_t$-parameters $\lambda_t$ and $\mu_t$ where $\lambda_t$ and $\mu_t$ are bounded. Define $p_n(t)$, $\lambda_n(t)$, and $\mu_n(t)$ by

$$\begin{aligned} p_n(t) &= P[Q_t = n], \\ \lambda_n(t) &= E[\lambda_t \mid Q_t = n], \\ \mu_n(t) &= E[\mu_t \mid Q_t = n]. \end{aligned} \tag{3.8}$$

# Gyöngi's Theorem: The Itô Counterpart

Given an Itô process $X$

$$dX_t = a_t dt + b_t dW_t$$

There exists a Markov process

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$$

with the same marginals,

$$\mu(x,t) = E[a_t \mid X_t = x], \quad \sigma^2(x,t) = E[b_t^2 \mid X_t = x]$$

if $a, b$ bounded and $b$ bounded away from zero

# But the Restrictions are Important…

The Heston model

$$\frac{dX_t}{X_t} = r\,dt + \sqrt{V_t}\,dW_t^1$$

$$dV_t = \kappa(\theta - V_t)\,dt + \nu\sqrt{V_t}\,dW_t^2$$

violates upper and lower bound conditions

Steve’s student Gerard Brunick has significantly extended Gyöngi’s Theorem to cover such cases.

# Even Simpler Example…

The mixture model:

**At $t$=0**

**1/2**

$$\frac{dX_t}{X_t} = r\,dt + \sigma_0 dW_t$$

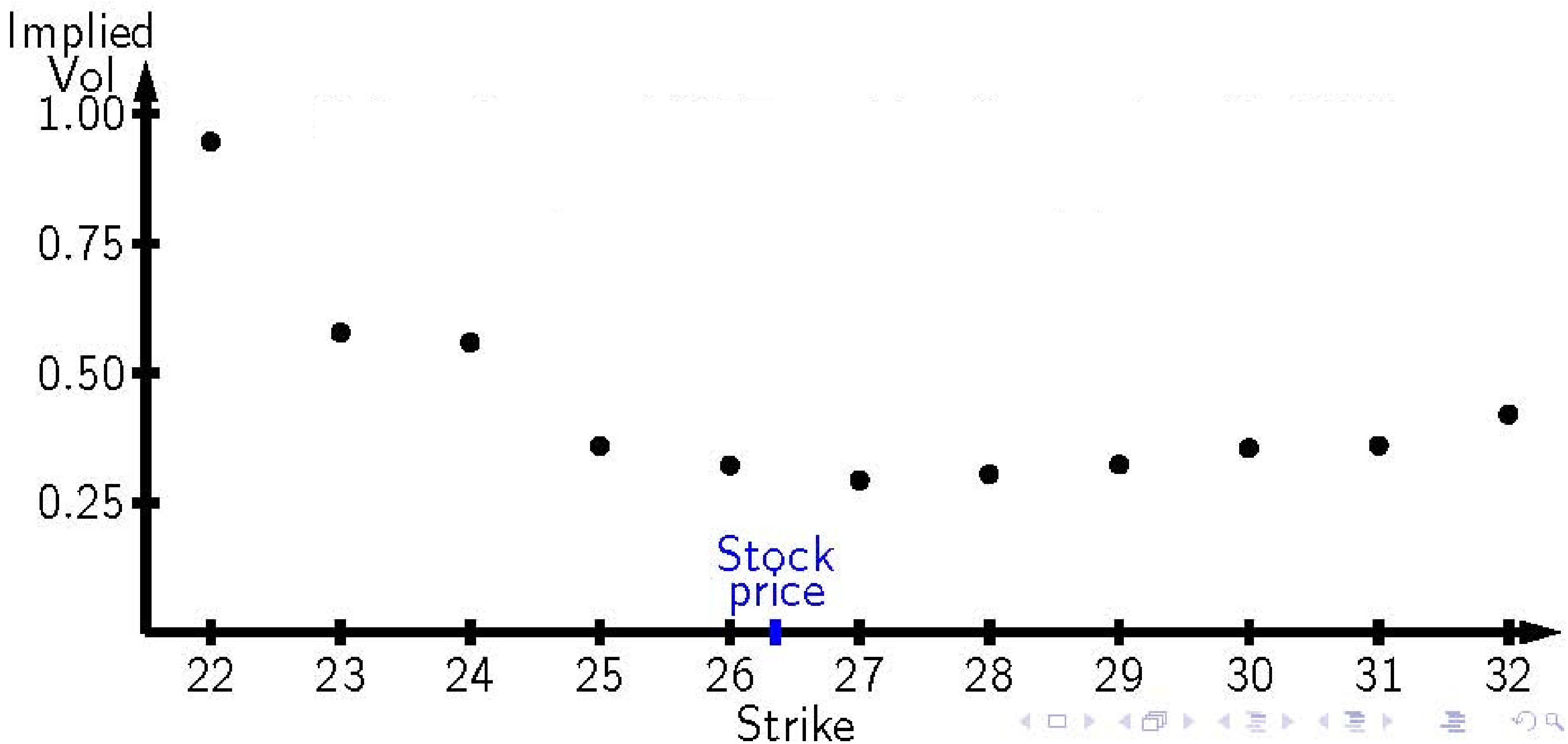
**1/2**

$$\frac{dX_t}{X_t} = r\,dt + \sigma_1 dW_t$$

An aside: PG and Ken Kim represent the Heston model as an infinite gamma mixture of lognormals and use this for exact simulation

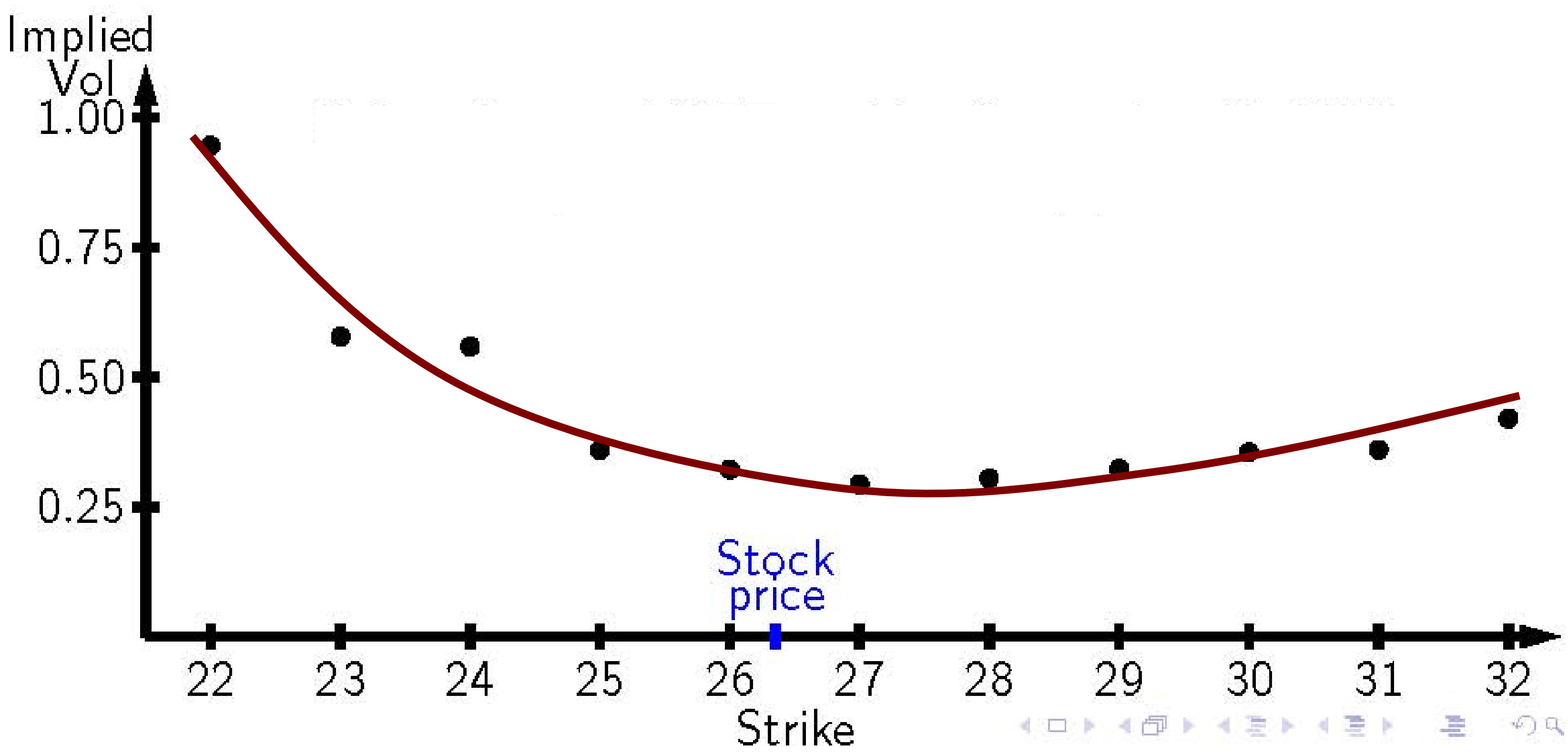
# Why Do We Care About Marginals?

We observe implied volatilities (prices) at multiple strikes…



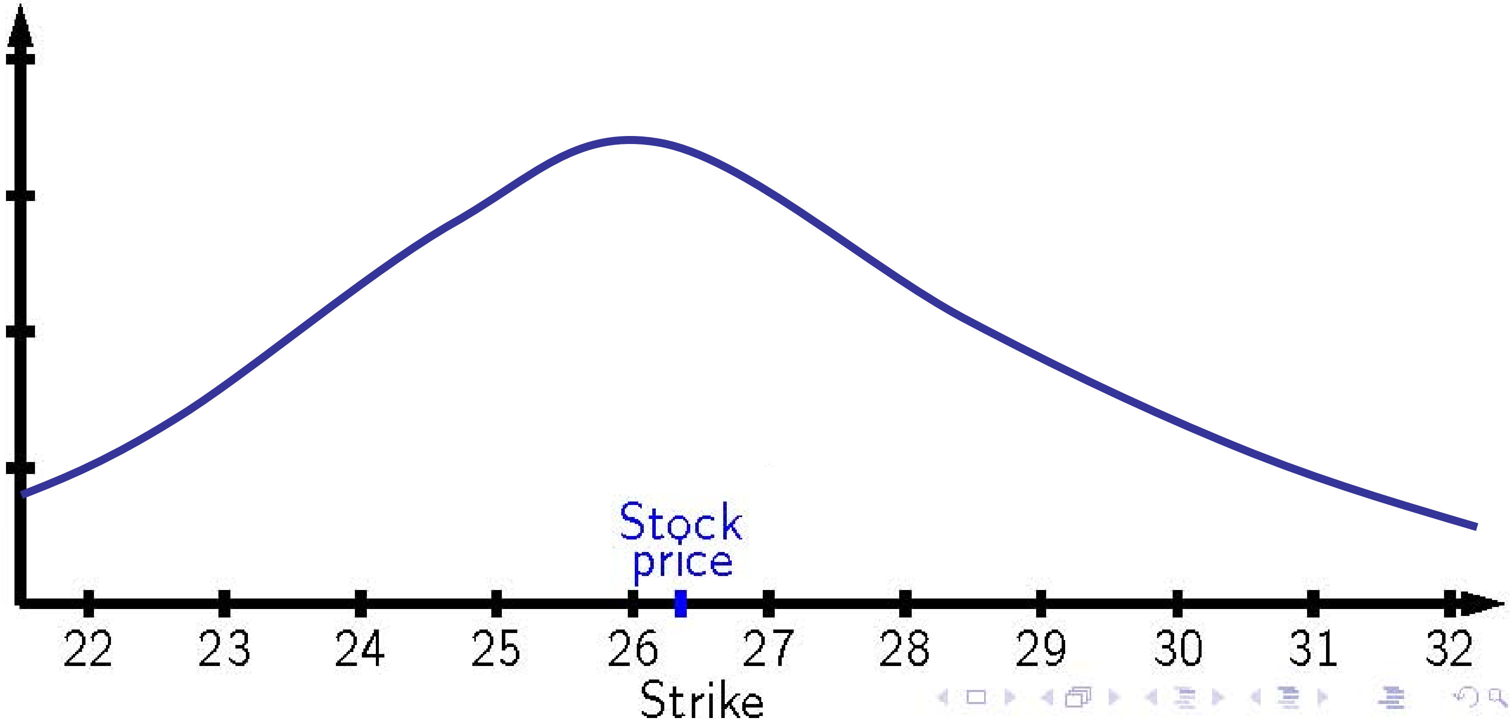
# Why Do We Care About Marginals?

Observing prices at all strikes…

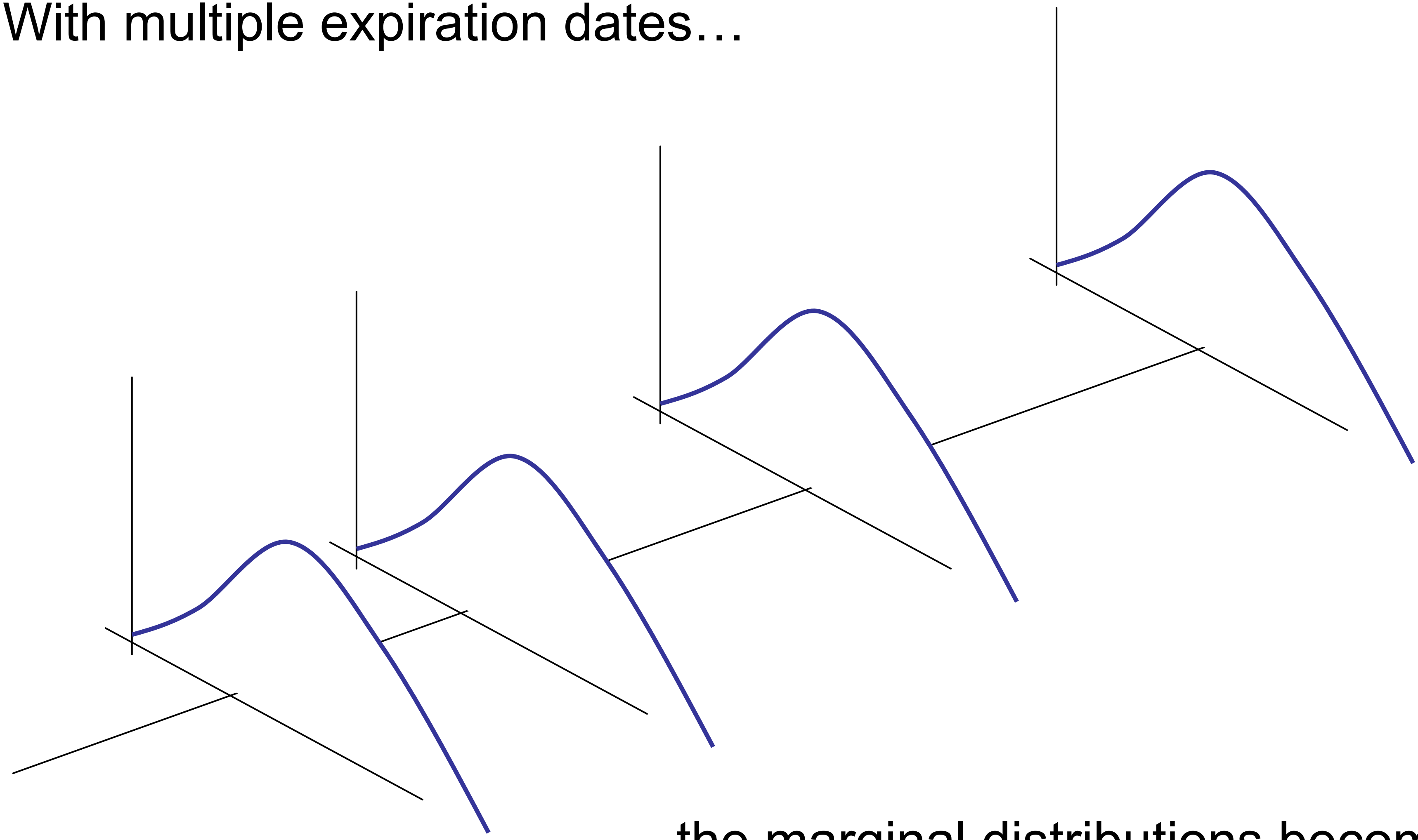


# Why Do We Care About Marginals?

…is equivalent to observing the underlying density at expiration



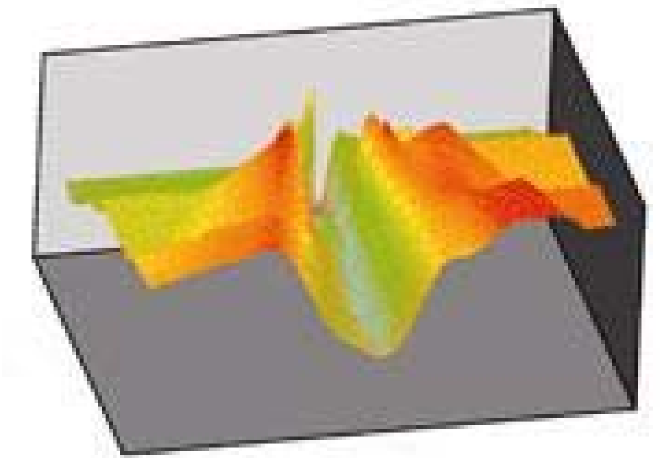
With multiple expiration dates…



…the marginal distributions become the key input to modeling

# Dupire's Local Volatility: Practitioner's Gyöngi

Given a set of marginals, construct a local volatility surface



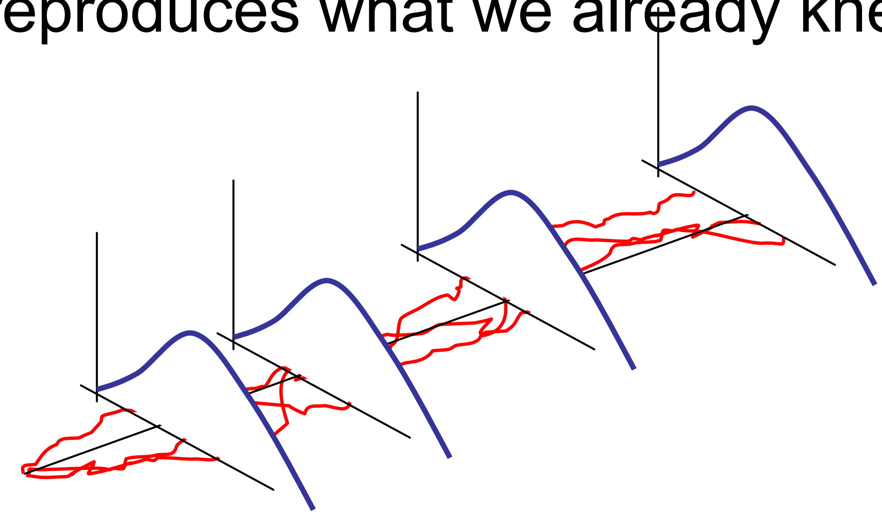
so that

$$\frac{dS_t}{S_t} = r\,dt + \sigma(S_t, t)\,dW_t$$

matches the marginals

# Is this Circular?  Do Dynamics Matter?

The model reproduces what we already knew (marginals)

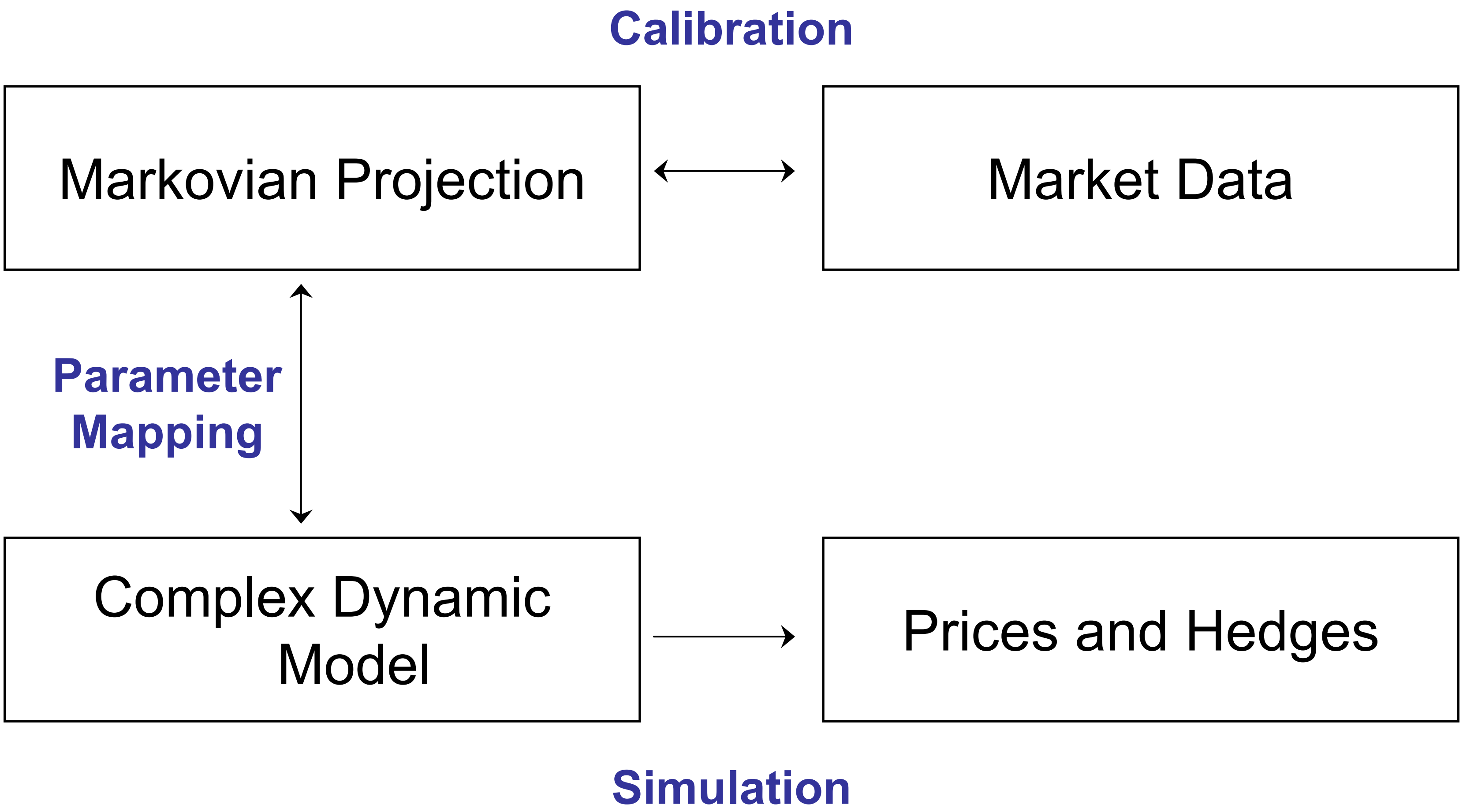


But it can be used to price and hedge path-dependent options

Will this work?

Wrong dynamics → poor hedging performance

# Putting the Pieces Together



# To Conclude…

The line of work Steve has presented is

- fundamental to theory (and difficult!)
- important to practice