Differential Equation Limits for Sparse Algorithms

INFORMS 2006 Markov Lecture Discussant:

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Random Process Analogues of Two Basic Limit Theorems

	$1/\sqrt{N}$ SCALING	$1/N$ SCALING
Sum of N random variables	Central limit theorem	Large deviation bound
Random process, scale parameter N	Diffusion Limit ▪ Ethier/Kurtz, '86 ▪ Jacod/Shiryaev, '03	Exponential rate of convergence to ODE

Markov Process Dynamics

COMPENSATOR
= DRIFT

MARKOV JUMP PROCESS
IN OPEN SET $U \subset R^d$

MARTINGALE SUBJECT
TO EXPONENTIAL INEQUALITY

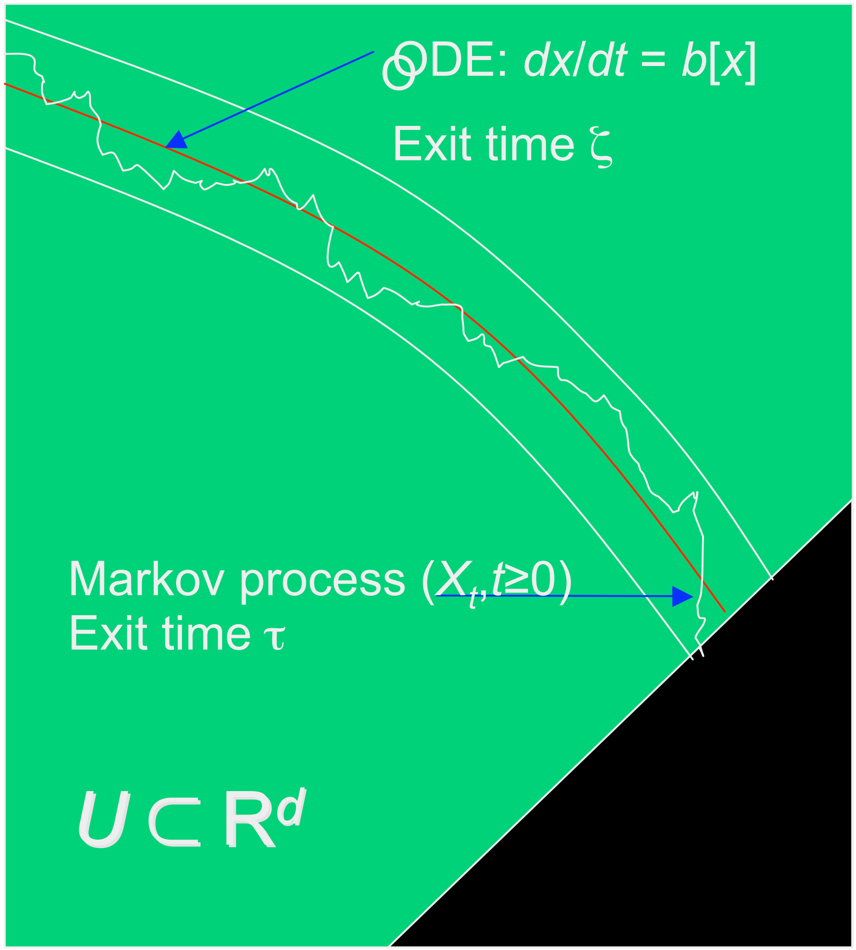
Limiting Differential Equation

$$x_t = x_0 + \int_0^t b[x_s]\,ds$$

DETERMINISTIC
TRAJECTORY

LIPSCHITZ VECTOR FIELD
ON OPEN SET $U \subset R^d$

Exit Probability from a Tube



- Jump size $O(1/N)$
- Jump rate $O(N)$
- Suppose ODE trajectory not tangential to boundary.
- Given bounds on β-b, & on exponential moments of M, both

$$\mathbf{P}[\sup_{t \le T} \|X_t - x_t\| > \varepsilon],$$

&

$$\mathbf{P}[\|X_\tau - x_\zeta\| > \varepsilon]$$

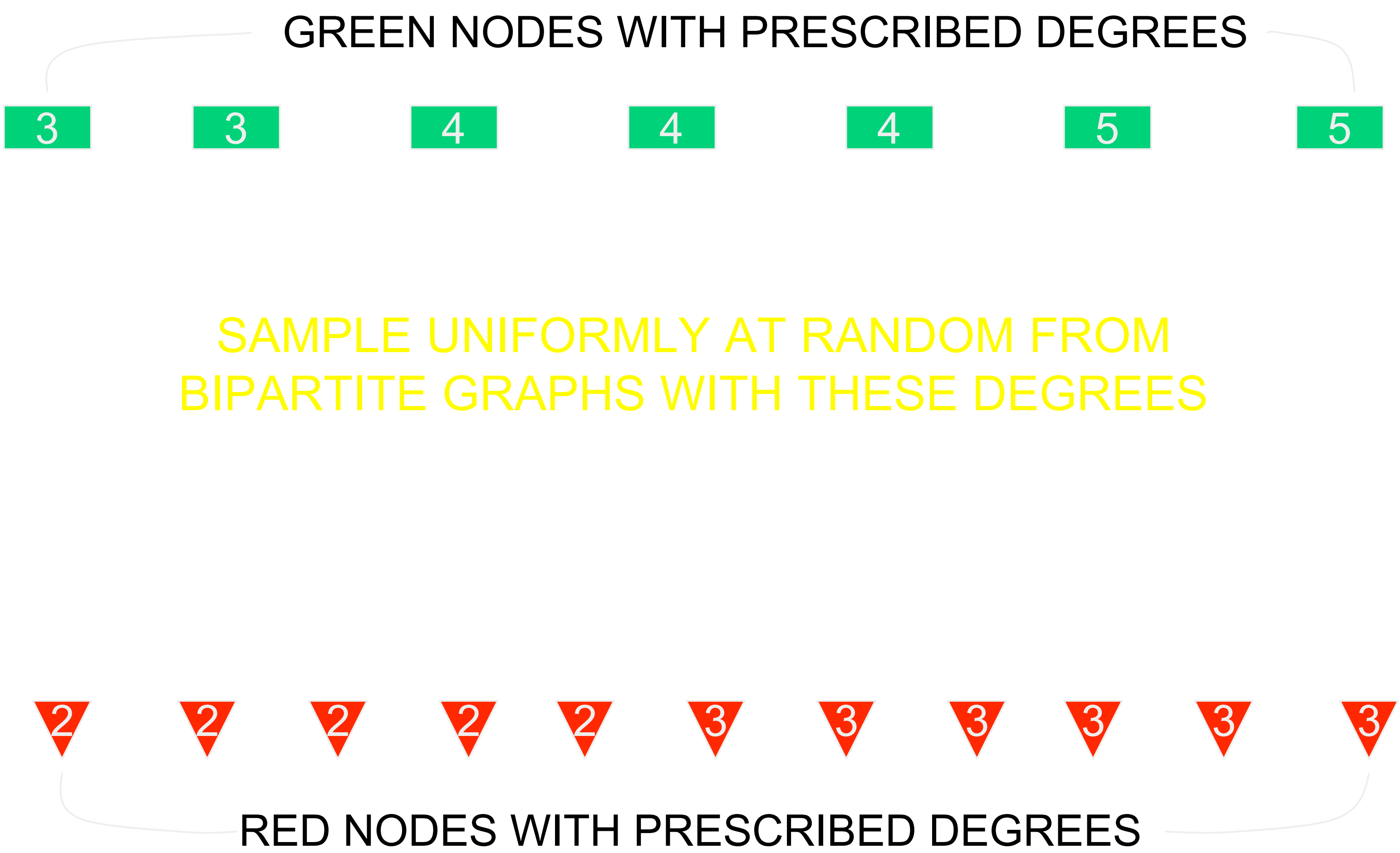
decay as $c_0 \exp[-cN]$, for computable $c>0$.

Models for Combinatorial Algorithms

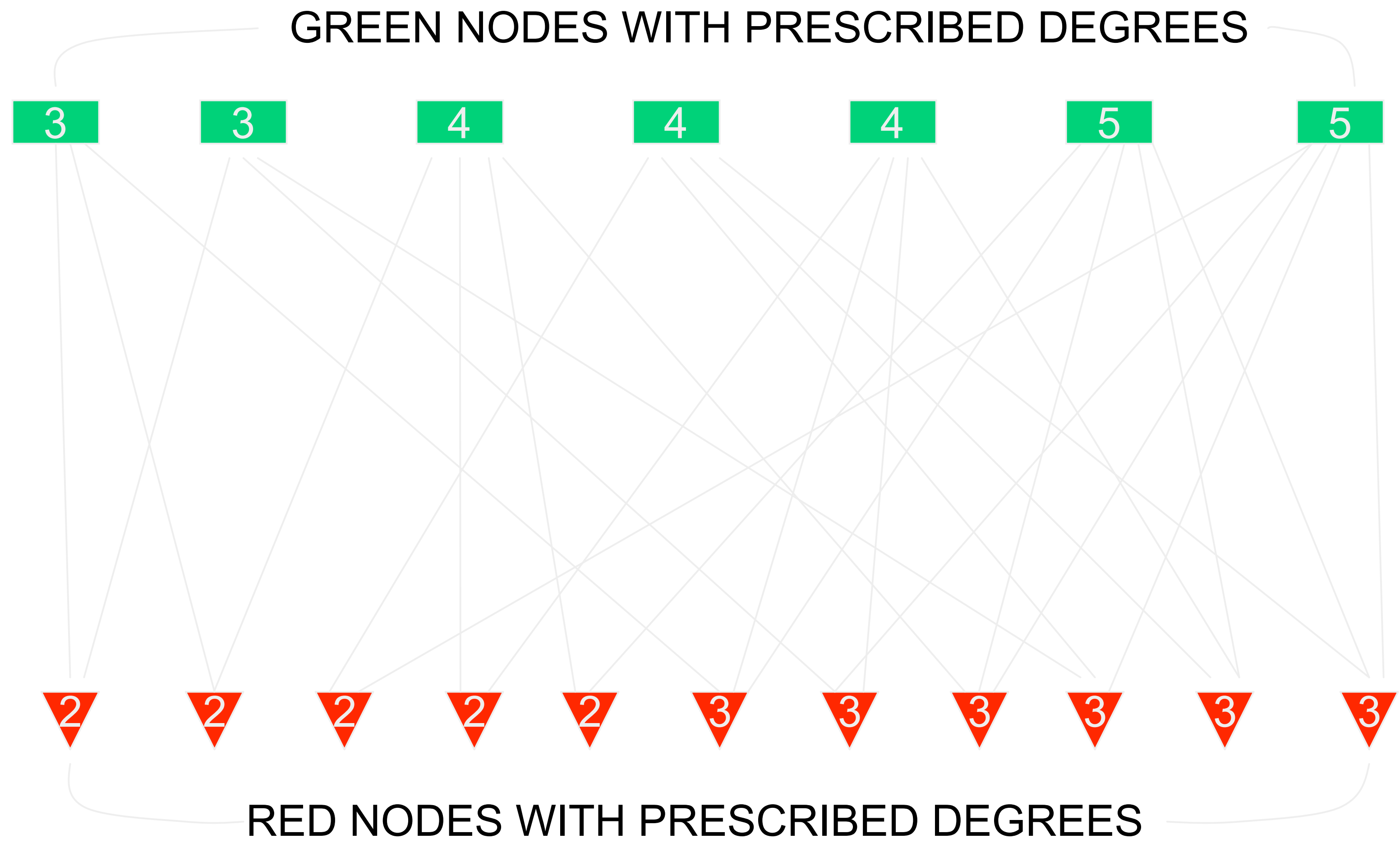
- "I can simulate a combinatorial algorithm. Why build an analytic model?"

1. The ODE has parameters which reveal analytically where phase transitions occur between solvable and unsolvable problems.
2. A Markov model makes explicit the statistical assumptions about the problem's structure.
3. Predict behavior in cases too large to simulate.

Multipurpose Modeling Tool: Random Bipartite Graphs



Random Bipartite Graph:
Realization Under Degree Constraints



Bipartite Graph Interpretations

	GREEN NODE i	RED NODE k	EDGE $\{i, k\}$
LT Codes	Source symbol i	Coded symbol k	$i \in$ {inputs for k}
Boolean satisfiability	Boolean variable i	Logical clause k	$i \in$ {inputs for k}
Matching problems	Job i	Server k	k is able to perform i

Factor Graph Abstraction

	GREEN NODE *i*	RED NODE *k*	EDGE $\{i, k\}$
Hypergraph	Vertex $i \in V$	$E_k \subset V$ "hyperedge"	$i \in E_k$ "incidence"
Factor Graph ▪ Aji &McEliece '00 ▪ Kschischang, Loeliger &Frey '01	Variable *i*	Function f_k	{arguments of f_k} $\ni i$

Examples of Algorithms Leading to Finite-Dimensional Markov Chains

1. Bipartite matching – Karp & Sipser ’81

2. Random Walk Sat – Schöning ‘99

3. LT decoding via Hypergraph 2-Core –

   Luby ’00; Maneva & Shokrollahi ’06; Hajek ’06

Hypergraph 2-Core

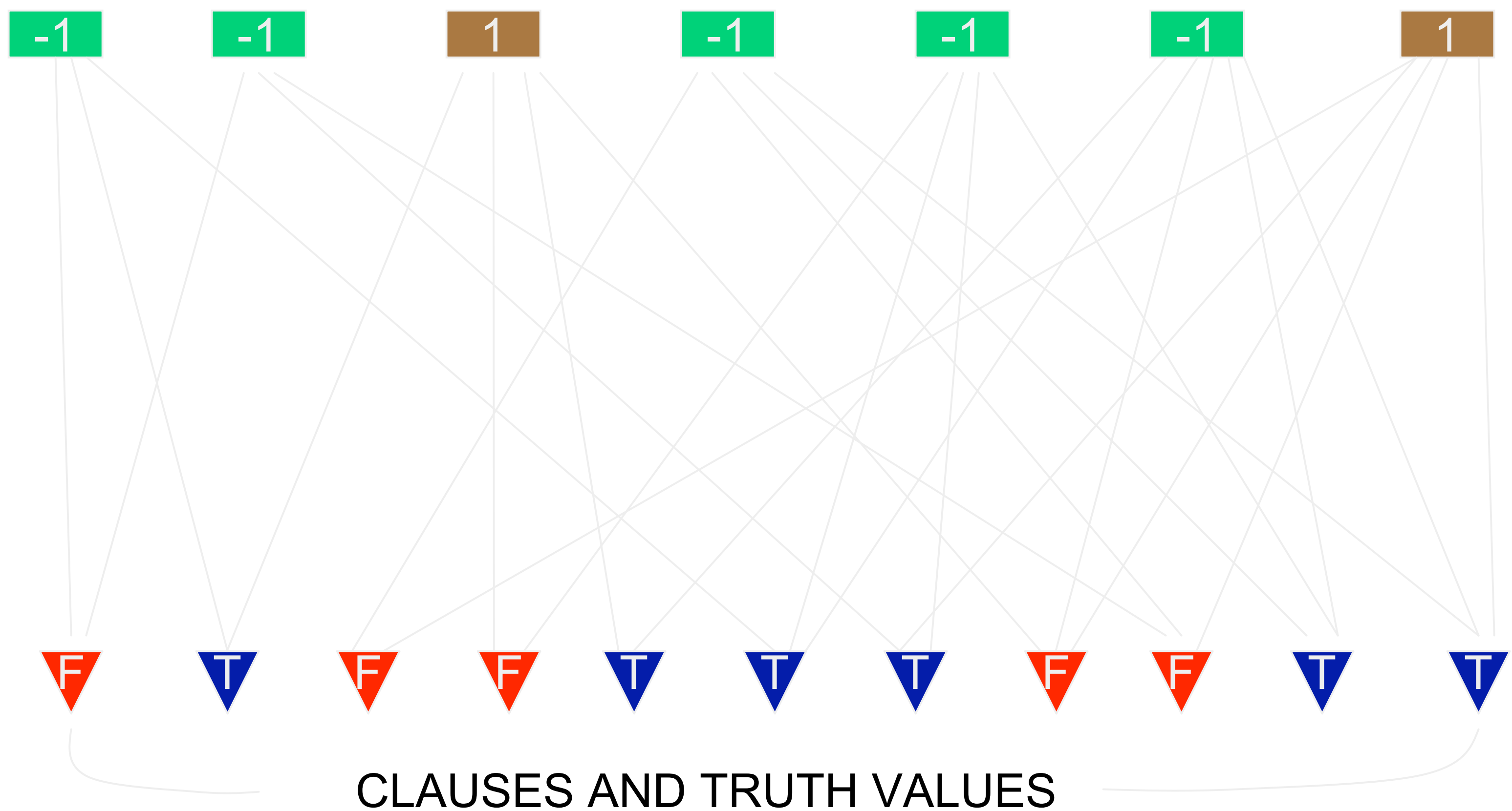
Select at random an incidence touching a degree 1 vertex, if one exists; remove entire hyperedge.

Simplifying Principle (Core)

- Call a vertex of degree ≥2 "heavy".
- After each hyperedge removal, restriction of hypergraph to the heavy vertices is still sampled uniformly, given its vertex degrees & edge weights.
- Hence a vector of vertex degrees, and a vector of edge weights, suffice to define "state" of Markov process.

Random Walk SAT

Select at random an incidence in a FALSE clause; flip the ±1 value of the variable



Simplifying Principle (SAT)

- $X^{c,i}$:= |clauses with *c* correctly set, and *i* incorrectly set variables|.
- $Y^{t,f}$:= |variables appearing in *t* TRUE and *f* FALSE clauses|.
- Matrices ($X^{c,i}$) & ($Y^{t,f}$) are called marginals.
- If a SAT problem is uniformly distributed, given its marginals, then it remains so after each transition of Random Walk SAT.
- Hence ($X^{c,i}$) & ($Y^{t,f}$) serve as the state for a Markov process model of Random Walk SAT.

Further Reading

- Dimitris Achlioptas, *Lower bounds for random 3-SAT via differential equations*, '01.
- Luby, Mitzenmacher, Shokrollahi, *Analysis of random processes via* AND-OR *tree evaluation*, '98.
- R.W.R.D. & J.R. Norris: *Structure of large random hypergraphs*, '05.
- …, *Differential equation approximations for Markov chains* (Survey)* .
- …, *Cores & cycles in random hypergraphs*, I & II *.

* to be posted on ArXiv