MIXING MODELS TO CAPTURE STOCK PRICE VOLATILITY

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Outline

Black-Scholes-(Merton)

European call: Right to buy one unit of a risky asset at an *expiration time* T at a *strike price* K agreed upon today (time zero). The call pays $(S_T - K)^+ \triangleq \max\{S_T - K, 0\}$ at expiration.

The price today of the call should be

$$S_0 N(d_+) - e^{-rT} K N(d_-),$$

where

- S_0 is today's price of the risky asset,
- r is the continuously compounding rate of interest,
- N is the standard cumulative normal distribution function,
- $$d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} - \left(r \pm \frac{1}{2}\sigma^2\right)T\right].$$
- σ, which is positive, is the risky asset price "volatility."

What Black-Scholes-(Merton) teaches us

Preposterous result: The price of the European call does not directly depend on the expected rate of growth of the underlying asset price.

Insightful result: The *volatility* σ of the risky asset is the key parameter.

Derivation of Black-Scholes-(Merton)

Replicate the call payoff by trading.

The two assets for trading are:

- a *money market account* with constant rate of interest r (we assume $r = 0$),
- the *risky asset* with price satisfying

$$dS_t = \alpha_t S_t \, dt + \sigma S_t \, dW_t,$$

  where

  - W is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ relative to a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$,
  - α_t is an adapted process,
  - σ is a positive constant.

  Because α_t is not required to be constant, nor even deterministic, S_t is not assumed to be a geometric Brownian motion.

Portfolio

Create a portfolio of cash and the risky asset whose value at each time t is the value of the call at that time.

- X_t – Value of the portfolio at time t.
- Δ_t – Number of units of the risky asset held by the portfolio at time t.
- $X_t - \Delta_t S_t$ – Amount of cash held by the portfolio at time t.
- Evolution of portfolio value –

$$dX_t = \Delta_t \, dS_t, \quad 0 \leq t \leq T.$$

Matching evolutions

Let $c(t, S_t)$ denote the price of the call at time t when the underlying asset price is $S(t)$. Note: $c(T, s) = (s - K)^+$ for all $s > 0$.

Itô's formula:

$$\begin{aligned} dc(t, S_t) &= c_t(t, S_t)\, dt + c_s(t, S_t)\, dS_t + \frac{1}{2} c_{ss}(t, S_t) \underbrace{dS_t\, dS_t}_{\sigma^2 S_t^2\, dt} \\ &\stackrel{?}{=} dX_t \\ &= \Delta_t\, dS_t. \end{aligned}$$

Need $c_t(t, s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t, s) = 0$ for $0 \le t < T$ and $s > 0$. Black-Scholes-(Merton) formula provides the solution to this PDE.

Replication:
Start with $X_0 = c(0, S_0)$ and take $\Delta_t = c_s(t, S_t)$, $0 \le t \le T$.
Then $X_T = c(T, S_T) = (S_T - K)^+$.

Risk-neutral (martingale) probability measure[1,2]

We write

$$dS_t = \alpha_t S_t\, dt + \sigma S_t\, dW_t = \sigma S_t \left[\frac{\alpha_t}{\sigma}\, dt + dW_t \right] = \sigma S_t\, d\widetilde{W}_t,$$

where

$$\widetilde{W}_t = \int_0^t \frac{\alpha_u}{\sigma}\, du + W_t.$$

Using Girsanov's Theorem, we construct a new probability measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion. Under $\widetilde{\mathbb{P}}$, the risky asset price is a geometric Brownian motion and a martingale. $\widetilde{\mathbb{P}}$ is called the *risk-neutral measure*.

[1] Harrison, J. M. & Kreps, D. M. (1979) Martingales and arbitrage in multiperiod security markets, *J. Econom. Theory* **20**, 381–408

[2] Harrison, J. M. & Pliska, S. R. (1981) Martingales and stochastic integrals in the theory of continuous trading, *Stoch. Proc. Appl.* **11**, 215–260.

Risk-neutral pricing

$$dS_t = \sigma S_t \, d\widetilde{W}_t.$$

For $0 \le t \le T$ and $s > 0$, define

$$f(t,s) = \widetilde{\mathbb{E}}\big[(S_T - K)^+ \big| S_t = s\big].$$

Then

$$f(T,s) = (s-K)^+ = c(T,s), \quad s > 0.$$

The Kolmogorov backward equation for $f(t,s)$ is

$$f_t(t,s) + \frac{1}{2}\sigma^2 s^2 f_{ss}(t,s) = 0.$$

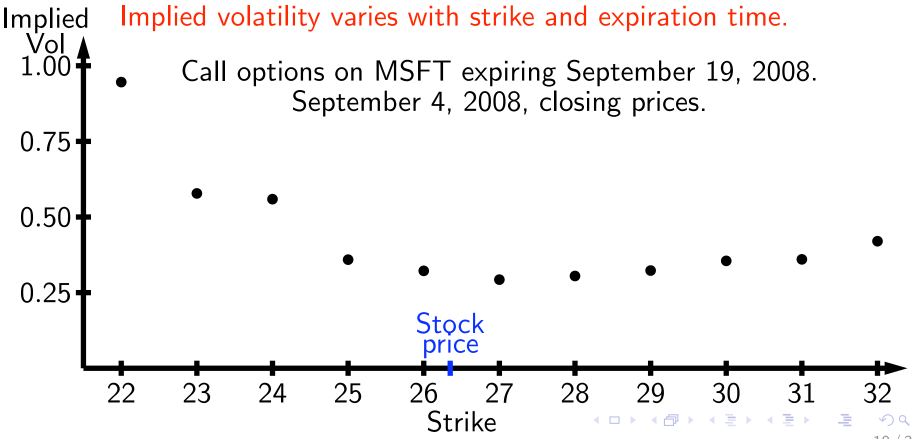
In other words, $f(t,s) = c(t,s)$.

Risk-neutral pricing formula:

$$c(t,s) = \widetilde{\mathbb{E}}\big[(S_T - K)^+ \big| S_t = s\big].$$

Criticism of Black-Scholes-(Merton)

Fix an underlying asset S. There is no single volatility parameter σ that causes the Black-Scholes-(Merton) formula to give the correct (market) price of options written on that asset with a variety of different strikes and expiration times.



Why does this matter?

Pricing – We want a model that we can calibrate to options that are traded and then use to price options that are not traded.

Vanilla options by interpolation – If we know the prices (and hence the implied volatilites) for calls with strikes K_0 and K_2 and want to price a call with strike $K_1 \in (K_0, K_2)$, we can interpolate the implied volatilities and use Black-Scholes-(Merton) with the interpolated volatility parameter.

Exotic options – Suppose we want to price a knock-out barrier option, which pays

$$(S_T - K)^+ I_{\{M_T \leq B\}},$$

where

$$M_T \triangleq \max_{0 \leq t \leq T} S_t.$$

Now what volatility should we use?

Why does this matter?

Hedging

- Hedge a book of options. Negative of replication.
- Want total portfolio (book plus hedge) to be insensitive to price changes.
- Requires a single model for the underlying asset price.

Four ways to proceed

1. Mixture of Black-Scholes-(Merton) models
2. Stochastic volatility models
3. Implied volatility models
4. Local volatility model

Mixture of Black-Scholes-(Merton) models

Recall $dS_t = \sigma S_t \, d\widetilde{W}_t$, so

$$S_T = S_0 \exp\left\{\sigma \widetilde{W}_T - \frac{1}{2}\sigma^2 T\right\}.$$

Let σ be a random variable, taking either the positive value c_1 or the positive value c_2. Then the distribution of S_T is a mixture of log-normals.

- For fixed T, mixture model fits well to call and put prices.
- Easy to implement.
- Not a reasonable dynamic model. Once σ has been realized, it will immediately be observed, and the mixture property is lost.
- No good for hedging.

Four ways to proceed

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Stochastic volatility models

Example (Heston[3])

$$
\begin{aligned}
dS_t &= \alpha_t S_t\, dt + \sqrt{v_t}\, S_t\, dW_t^1, \\
dv_t &= \lambda\left(\overline{v} - v_t\right) dt + \beta\sqrt{v_t}\, dW_t^2, \\
\langle W^1, W^2\rangle_t &= \rho t.
\end{aligned}
$$

Closed-form formulas for Fourier transforms of option prices.

[3] S. Heston (1993) A closed-form solution for options with stochastic volatility, with applications to bond and currency options, *Review Financial Studies* **6**, 327–343.

Stochastic volatility models (continued)

Example (Fouque, Papanicolaou, Sircar[4])

$$
\begin{aligned}
dS_t &= \alpha_t S_t \, dt + e^{Y_t} S_t \, dW_t^1, \\
dY_t &= \frac{\lambda}{\varepsilon^2}\left(\overline{Y} - Y_t\right) dt + \frac{\beta}{\varepsilon} \, dW_t^2, \\
\langle W^1, W^2 \rangle_t &= \rho t.
\end{aligned}
$$

Asymptotic expansion of call prices in powers of ε, with zero-order term given by Black-Scholes-(Merton) formula.

[4] J.-P. Fouque, G. Papanicolaou and R. Sircar (2000) *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press.

Comments on stochastic volatility models

Calibration:

- First choose parameters, and then “run” the model to output option prices.
- Compare model outputs with market prices. Try to choose better parameters, typically to reduce the sum of squared errors. This is a nonlinear optimization problem in which each function evaluation is expensive.
- Subsequent recalibrations result in paper profits or losses and upset hedges.

Four ways to proceed

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Implied volatility models

Start with $c(0, S_0; t, K)$ for $0 < t \leq T$ and $K > 0$. For each t and K, define the *implied volatility* $\sigma(t, K)$ to be the solution of the equation

$$c(0, S_0; t, K) = \widetilde{\mathbb{E}}\left[\left(\underbrace{S_0 \exp\left\{\sigma \widetilde{W}(t) - \frac{1}{2}\sigma^2 t\right\}}_{S_t} - K\right)^+\right].$$

Obtain thereby a *implied volatility surface*

$$\sigma(t, K), \quad 0 < t \leq T, \; K > 0.$$

Build a stochastic model to evolve the implied volatility surface forward.

In contrast to stochastic volatility models, the initial market call prices are an input, not an output, of implied volatility models.

Arbitrage problem for implied volatility models

- Assume that today $\sigma(t, K) = 0.20$ for all t and K. This is a *flat implied volatility surface*. (Black-Scholes-(Merton) assumes the implied volatility surface is flat.)
- Suppose that tomorrow we will have another flat implied volatility surface, at either 0.25 or 0.15. Today we do not know which scenario will occur.
- At time zero, buy Δ_i calls with strike K_i, $i = 1, 2, 3$.
- Choose non-zero Δ_1, Δ_2 and Δ_3 so that the portfolio has value zero tomorrow, regardless of which scenario occurs. This requires that we solve two homogeneous equations in three unknowns.
- Price of portfolio must be zero today, or else there is an arbitrage.

Arbitrage problem for implied volatility models

- A necessary and sufficient condition to rule out arbitrage is known.[5,6]
- It is so complex that it has thus far prevented the implementation of such a model.

[5] P. Schönbucher, A market model for stochastic implied volatility, Working paper, May 1999.

[6] M. Schweizer and J. Wissel (2008), Term structure of implied volatilities: absence of arbitrage and existence results, *Math. Finance* **18**, 77-114.

Four ways to proceed

1. Mixture of Black-Scholes-(Merton) models
2. Stochastic volatility models
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4. Local volatility model

Local volatility model – Implying risk-neutral densities from call prices.

Suppose we are given $c(0, S_0; T, K)$ for all $K > 0$. We can imply the risk-neutral densities $p(T, s) = p(0, S_0; T, s)$ from the formula

$$
\begin{aligned}
c\big(0, S_0; T, K\big) &= \widetilde{\mathbb{E}}\big[\big(S_T - K\big)^+ \big| S_0\big] \\
&= \int_K^\infty (s - K) p(T, s)\, ds.
\end{aligned}
$$

Differentiate once:

$$
\begin{aligned}
\frac{\partial}{\partial K} c\big(0, S_0; T, K\big) &= -(K - K) p(T, K) - \int_K^\infty p(T, s)\, ds \\
&= -\int_K^\infty p(T, s)\, ds.
\end{aligned}
$$

Differentiate again:

$$
\frac{\partial^2}{\partial K^2} c\big(0, S(0); T, K\big) = p(T, K).
$$

Implying the local volatility surface.[7,8]

Assume the evolution of the risky asset price:

$$dS_t \;=\; \sigma\big(t, S_t\big) S_t \, d\widetilde{W}_t = \gamma\big(t, S_t\big)\, d\widetilde{W}_t, \quad 0 \le t \le T.$$

Forward equation satisfied by transition density:

$$\frac{\partial}{\partial T} p(0, S_0; T, s) = \frac{1}{2}\frac{\partial^2}{\partial s^2}\Big(\gamma^2(T, s) p(0, S_0; T, s)\Big).$$

Again we set $p(T, s) = p(0, S_0; T, s)$. Then

$$\begin{aligned} \frac{\partial}{\partial T} c\big(0, S(0); T, K\big) &= \int_K^\infty (s - K) \frac{\partial}{\partial T} p(T, s)\, du \\ &= \frac{1}{2}\int_K^\infty (s - K) \frac{\partial^2}{\partial s^2}\Big(\gamma^2(T, s) p(T, s)\Big)\, ds. \end{aligned}$$

[7] B. Dupire (1994) Pricing with a smile, *Risk* **7**, 18–20.

[8] E. Derman and I. Kani (1994) Riding on a smile, *Risk* **7**, 32–39

Implying the local volatility surface.

Integrate by parts:

$$
\begin{aligned}
\frac{\partial}{\partial T} c\big(0, S(0); T, K\big) &= \frac{1}{2}\int_K^\infty (s-K)\frac{\partial^2}{\partial s^2}\Big(\gamma^2(T,s)p(T,s)\Big)\, ds \\
&= -\frac{1}{2}\int_K^\infty \frac{\partial}{\partial s}\Big(\gamma^2(T,s)p(T,s)\Big)\, ds \\
&= \frac{1}{2}\gamma^2(T,K)p(T,K).
\end{aligned}
$$

Recall

$$
p(T,K) = \frac{\partial^2}{\partial K^2} c\big(0, S(0); T, K\big).
$$

Solve for the function $\gamma^2(T,K)$ for T in some range and $K > 0$.
Recall $\sigma(T,K)K = \gamma(t,K)$.

Local volatility model – Practical issues

- Local volatility model takes call prices as inputs.
- There is no freedom to calibrate to exotic options.
- Volatility surface not stable over time.

Local volatility model – Theoretical issues

Theorem (First Fundamental Theorem of Asset Pricing[9])

Assume

- *underlying asset is a semimartingale,*
- *call price processes are semimartingales,*
- *there is no free lunch with vanishing risk (essentially, no arbitrage).*

Then there exists a risk-neutral probability measure $\widetilde{\mathbb{P}}$ *under which all these price processes are local martingales.*

Under mild additional conditions, then S has the representation

$$dS_t = \gamma_t \cdot d\widetilde{W}_t,$$

where γ_t is a vector-valued adapted process and $\widetilde{W}_t$ is a vector of independent Brownian motions under $\widetilde{\mathbb{P}}$.

[9] F. Delbaen and W. Schachermayer (1994) A general version of the fundamental theorem of asset pricing, *Math. Annalen* **300**, 463–520.

Local volatility model – Theoretical issues

Assume

$$dS_t = \gamma_t \cdot d\widetilde{W}_t.$$

Theorem (Krylov[10], Gyöngy[11])

Assume that the the largest eigenvalue of $\gamma_t \gamma_t^{tr}$ is bounded and the smallest eigenvalue is bounded away from zero, uniformly in t and ω. Then there is a volatility surface $\sigma(t,s)$ such that

- $dS_t^{vs} = \sigma(t, S_t^{vs}) S_t^{vs} dW_t^{vs}$ *has a weak solution, unique in law;*
- *for each* $t \geq 0$*, we have* $\mathcal{L}(S_t) = \mathcal{L}(S_t^{vs})$*;*
- *prices of calls on S agree with prices of calls on S^{vs}.*

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[10] N. Krylov (1984) Once more about the connection between elliptic operators and Itô's stochastic equations, *Statistics and Control of Stochastic Processes, Steklov Seminar*, 214–229.

[11] I. Gyöngy (1986) Mimicking the one-dimensional marginal distributions of processes having an Itô differential, *Prob. Theory and Related Fields* **71**, 501–516.

Local volatility model – Mixture example

Assume $\widetilde{\mathbb{P}}\{\sigma_0 = c_1\} = \widetilde{\mathbb{P}}\{\sigma_0 = c_2\}$, where $c_2 > c_1 > 0$. Assume

$$\sigma_t = \sigma_0 \text{ so } \gamma_t = \sigma_0 S_t, \quad 0 \leq t \leq T,$$

where

$$dS_t = \sigma_0 S_t \, d\widetilde{W}_t, \quad 0 \leq t \leq T.$$

"Smallest eigenvalue" of $\gamma_t^2 = \sigma_0 S_t^2$ is not bounded away from zero.

Nonetheless, there is a local volatility surface $\sigma(t, s)$, and the stochastic differential equation

$$dS_t^{vs} = \sigma(t, S_t^{vs}) S_t^{vs} \, dW_t^{vs}$$

has a unique strong solution.[12]

[12] D. Brigo and F. Mercurio (2002) Lognormal-mixture dynamics and calibration to market volatility smiles, *Internat. J. Theoret. Appl. Finance* **5**, 427–446.

Local volatility model – Mixture example

- $dS_t = \sigma_0 S_t \, d\widetilde{W}_t$ – choose a volatility at time zero and use it throughout.
- $dS_t^{vs} = \sigma(t, S_t^{vs}) S_t^{vs} \, dW_t^{vs}$ – time-varying, state-dependent volatility.
- If $S_0 = S_0^{vs}$, then S_t and S_t^{vs} have the same one-dimensional distributions for all $t \geq 0$, so ...
- prices of calls on S agree with prices of calls on S^{vs}.
- S^{vs} makes sense as a dynamic model. S does not.

Dynamic mixture model - Outline of a general construction

Choose a sequence of partitions

$$\Pi_n : 0 = T_0 < T_1 < T_2 < \cdots < T_n < \infty.$$

Flip a coin at time T_0 to choose a volatility σ_0. If we use this volatility throughout, we obtain the process S.

Use S to construct a second process S^{Π_n} as follows. At time T_i, compute $\widetilde{\mathbb{P}}\{\sigma_0 = c_i | S_{T_i} = s\}$. Redraw the volatility with this distribution and use it on $[T_i, T_{i+1})$.

- For each t, S_t and $S_t^{\Pi_n}$ have the same distribution.
- Immediately after each T_i, observation of S^{Π_n} reveals the volatility being used on $[T_i, T_{i+1})$, but not the volatility that will be chosen at T_{i+1}.
- As $n \to \infty$, S^{Π_n} converges weakly to some S^{vs} in $C[0, \infty)$.
- $dS_t^{vs} = \sigma(t, S_t^{vs}) S_t^{vs} dW_t^{vs}$, where $(\sigma^{vs})^2(t, s) = \widetilde{\mathbb{E}}\big[\sigma_0^2 \big| S_t = s\big]$.

Extension to barrier options

Theorem (Brunick[13])

Assume

$$dS_t = \gamma_t \cdot d\widetilde{W}_t,$$

and define

$$M_t \triangleq \max_{0 \le u \le t} S_u.$$

Then there exists a function $\gamma(t, s, m)$ *and a weak solution to the stochastic differential equation*

$$dS_t^{vs} = \gamma(t, S_t^{vs}, M_t^{vs})\, dW_t^{vs},$$

where

$$M_t^{vs} \triangleq \max_{0 \le u \le t} S_u^{vs},$$

such that for each $t \ge 0$, *we have* $\mathcal{L}(S_t, M_t) = \mathcal{L}(S_t^{vs}, M_t^{vs})$.

[13] G. Brunick (2008) A weak existence result with application to the financial engineer's calibration problem, Ph.D. dissertation, Carnegie Mellon University.

Further discussion

Comments:

- $\gamma^2(t, s, m) = \widetilde{\mathbb{E}}\left[\gamma_t^2 \middle| S_t = s, M_t = m\right]$
- The stated theorem is a corollary of a more general result.
- At the level of generality given here, there can be multiple weak solutions to

$$dS_t^{vs} = \gamma(t, S_t^{vs}, M_t^{vs})\, dW_t^{vs}.$$

Some open problems:

- Obtain conditions under which uniqueness holds.
- Obtain a formula for $\gamma(t, s, m)$ in terms of prices of calls and barrier options.

Conclusions

Two observations about the development of a new generation of models for option pricing and hedging.

- Critical importance for the practice of finance.
- Source of fascinating new problems in mathematics.