
Robust Congestion Pricing Under Demand Uncertainty

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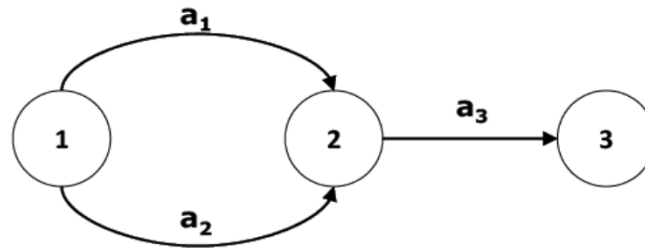
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Introduction

- After the initial idea of road pricing by Pigou (1920), the literature on congestion pricing is growing rapidly in theory and practice.
- Congestion pricing has been regarded as an efficient method to manage a congested traffic network
 - First-best pricing problem: marginal cost over all arcs in a network
 - Second-best pricing problem: a subset of arcs can be tolled, more practical
- A second-best pricing problem can be formulated as MPEC (Mathematical problem with equilibrium constraint), which is non-convex.
 - Upper level : min travel cost or max revenue
 - Lower level : user equilibrium network flow
- In this presentation, we investigate a *robust congestion (static and dynamic) pricing problem under demand uncertainty*.

Motivation: Two-route Problems

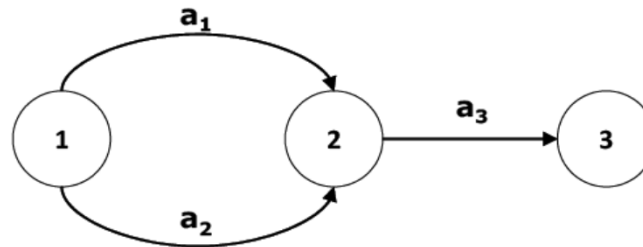


index (i)	Arc	from	to	unit cost C_a
1	a1	1	2	$\alpha_1 + \beta_1 f_1 + y$
2	a2	1	2	$\alpha_2 + \beta_2 f_2$
3	a3	2	3	$\alpha_3 + \beta_3 f_3$

Total Demand from node 1 to 3 : Q

- A classical example of the second best problem (Levy-Lambert, 1968; Marchand, 1968; Verhoef et al, 1996)
- One tolled route and one untolled alternative route are available

Motivation



- 2 competing congested routes

- Unit arc cost

$$c_{a_1} = 1 + 2f_1 + y$$

$$c_{a_2} = 2 + f_2$$

$$c_{a_3} = 1 + f_3$$

- Revenue maximization problem

$$\max_y yf_1$$

s.t

f = UE flow

$$y \geq 0$$

Motivation

- Optimal solution (Deterministic Problem)

Total demand	Optimal toll	Flow on path 1	Flow on path 2	Total revenue
8	4.5	1.5	6.5	6.75
10	5.5	1.83	8.17	10.08

- Realized revenue from true demand (D^*)

Optimal toll	Total revenue ($D^* = 8$)	Total revenue ($D^* = 10$)
4.5	6.75	9.75
5.5	6.41	10.08

- How to find robust optimal tolls?

Literature Review

- Congestion Pricing Problem

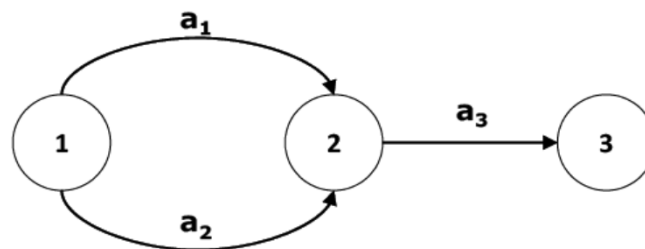
- Many studies on the **static** congestion pricing problem
 - e.g. Lawphongpanich and Hearn (2004)
 - Second-best Toll, MPEC approach and three equivalent formulations
- Limited attempts on the congestion pricing problem in **dynamic** transportation networks
 - Viti et al. (2003): Uniform toll with a simple grid search algorithm
 - Joksimovic et al. (2005) : Uniform and time-varying toll with a grid search algorithm
 - Wie (2007) : Triangular toll with a Hooke-Jeeves algorithm
 - Friesz et al. (2007) : Dynamic Optimal Toll Problem with Equilibrium Constraints (DOTPEC) problem based on Tan et al. (1979) reformulation

Literature Review

- (Static) Congestion Pricing under Uncertainty

- Waller et al. (2001)
System performance can be negatively impacted when single fixed demand is used.
- Nagae and Akamatsu (2006)
A stochastic singular control problem for choosing tolls from two discrete value.
- Gardner et al. (2008)
Scenario based robust optimization for first-best tolls under demand uncertainty.
- Ban et al. (2009)
Robust optimization for risk-averse second best toll pricing with multiple user equilibrium solutions
- Lou et al. (2010)
Robust optimization to consider all possible boundedly rational user equilibrium

Robust Road Pricing with Reaction Function



index (i)	Arc	from	to	unit cost C_a
1	a1	1	2	$\alpha_1 + \beta_1 f_1 + y$
2	a2	1	2	$\alpha_2 + \beta_2 f_2$
3	a3	2	3	$\alpha_3 + \beta_3 f_3$

Total Demand from node 1 to 3 : Q

Revenue Maximization Problem: Deterministic Problem

- Revenue maximization problem

$$\max_y yf_1$$

s.t

$$f = UE \text{ flow}$$

$$y \geq 0$$

Revenue Maximization Problem: Deterministic Problem

- User Equilibrium Flow (Reaction function)

→ Assume that both paths are used.

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{-\alpha_1 + \alpha_2 - y + \beta_2 Q}{\beta_1 + \beta_2} \\ \frac{\alpha_1 - \alpha_2 + y + \beta_1 Q}{\beta_1 + \beta_2} \\ Q \end{bmatrix}$$

Revenue Maximization Problem: Deterministic Problem

- Revenue Maximization problem

$$\begin{aligned} & \max_{y>0} yf_1 \\ & = \frac{-1}{\beta_1 + \beta_2} (y^2 + (\alpha_1 - \alpha_2 - \beta_2 Q)y) \end{aligned}$$

$$y^* = \begin{cases} (\beta_2 Q - \alpha_1 + \alpha_2)/2 & \text{if } Q > (\alpha_1 - \alpha_2)/\beta_2 \\ 0 & \text{if } Q \leq (\alpha_1 - \alpha_2)/\beta_2 \end{cases}$$

→ Optimal toll price is a function of total demand

Revenue Maximization Problem under uncertainty

- Box Uncertainty Set

$$Q \in U_Q = [\bar{Q}(1 - \theta), \bar{Q}(1 + \theta)]$$

- Robust Counterpart

$$\max_{z, y} z$$

$$\frac{-1}{\beta_1 + \beta_2} (y^2 + (\alpha_1 - \alpha_2 - \beta_2 Q)y) \geq z \quad Q \in U_Q$$

$$y_{LB} \leq y \leq y_{UB}$$

Revenue Maximization Problem under uncertainty

- The following relation for any real numbers u_i and v (Ben-Tal et al. 2004)

$$uQ \geq v \quad \forall Q \in [\bar{Q}(1-\theta), \bar{Q}(1+\theta)]$$
$$\Leftrightarrow (u - \theta |u|) \bar{Q} \leq v$$

- Computationally Tractable Robust Counterpart

$$\max_{z, y, \rho} z$$
$$\frac{-1}{\beta_1 + \beta_2} (y^2 + (\alpha_1 - \alpha_2)y) + \left(\frac{\beta_2 y}{\beta_1 + \beta_2} - \theta \rho \right) \bar{Q} \geq z$$
$$-\rho \leq \frac{\beta_2 y}{\beta_1 + \beta_2} \leq \rho$$
$$y_{LB} \leq y \leq y_{UB}$$

Numerical Experiments

- Objective value and optimal toll

θ	Objective Value		Optimal Toll	
	D	R	D	R
0.1	10.08	8.33	5.5	5
0.2	10.08	6.75	5.5	4.5
0.3	10.08	5.33	5.5	4

- Simulation Results (100 random demand)

θ	Worst case		Standard Deviation	
	D	R	D	R
0.1	8.27	8.35	1.00	0.91
0.2	6.45	6.78	2.01	1.64
0.3	4.64	5.37	3.02	2.19

General Cases

- However, in most cases, it is not easy to get the reaction function.
- In general, robust congestion pricing problem falls into the category of **robust non-convex optimization problem**.
- Unfortunately, most advances in RO have come from convex optimization problem. It is not known how to derive tractable robust counter part of non-convex (bi-level) optimization problem
- Some **heuristic approaches** for robust non-convex optimization problem
 - Yin and Lawphongpanich (2007): Cutting plane algorithm.
 - Bertsimas et al. (2009a): Robust local search (RLS) algorithm
 - Bertsimas and Nohadani (2010): Robust simulated annealing (RSA)

Dynamic User Equilibrium

Dynamic User Equilibrium (DUE) is one type of Dynamic Traffic Assignment (DTA) wherein the effective unit travel delay, including early and late arrival penalties, of travel for the same purpose is identical for all utilized path and departure time pairs

- Traffic Dynamic Models. Merchant and Nemhauser (1978a,b), Ran et al. (1993) and Ran and Boyce (1996)
- Friesz et. al (1993) proposed a variational inequality (VI) formulation that is a dynamic generalization of the static Wardropian user equilibrium
- Friesz et. al (2001) formulated DUE problem as differential variational inequality (DVI) with state-dependent time shift.
- Friesz et. al (2010) approximated dynamic network loading by using the second order Taylor expansion for computational efficiency.

Robust DOTPEC Problem

- Formulation: Dynamic Network

$$\min z = \sum_{p \in P} \int_{t_0}^{t_f} c_p(t, h(t)) h_p(t)$$

s.t.

$$\sum_{p \in P} \int_{t_0}^{t_f} \theta_p(t, h^*(t), y(t)) (h(t) - h^*(t)) \geq 0, \forall h \in \Omega$$

$$y_{LB} \leq y \leq y_{UB}$$

, where

$$\Omega = \left\{ h \geq 0; \frac{ds_w}{dt} = \sum_{p \in P_w} h_p(t), s_w(t_0) = 0, s_w(t_f) = Q_w, w \in W \right\}$$

$$Q \in U_Q = [\bar{Q}(1 - \theta), \bar{Q}(1 + \theta)]$$

Tolled effective path delay operator

$$\begin{aligned} \theta_p(t, h(t), y(t)) \\ = y_p(t) + c_p(t, h(t)) \end{aligned}$$

$$\begin{aligned} c_p(t, h(t)) \\ = D_p + F[t + D_p - T_A] \end{aligned}$$

Bi-level Cellular Particle Swarm Optimization

- Kennedy, J. & Eberhart, R. (1995)

Particle swarm optimization (PSO) maintains a swarm of agents, referred to as particles. These particles fly through the feasible region with each particle attracted by the best locations ever found by both the individual particle and by the whole swarm of the particles. The emergent behaviour of this system provides improvement to the objective function value.

- Shi, Y. et al. (2010)

Cellular Particle swarm optimization (CPSO) applies the notion of cellular automata to refine the flying trajectory of PSO. It outperforms all well-recognized PSO variants involved in their test, but there is no global convergence property.

- **Our Proposed Solution Approach**

Bi-level Cellular Particle Swarm Optimization (BCPSO) integrates a modified CPSO into a robust bi-level optimization framework extended from Bertsimas et al. 2009. In this modified CPSO, global convergence is guaranteed when iteration steps are infinite.

BCPSO Implementation

Initialization of parameters;

Initialize y_a , where $a \in A$;

Counter=0;

While outer termination criterion not satisfied

 Randomly initialize Q ;

 While inner termination criterion not satisfied;

 Calculate the user equilibrium with Q and y_a given;

 Update Q by one iteration of CPSO_{*i*} maximizing total travel cost;

 End while;

 Update y_a by one iteration of CPSO_{*o*} minimizing total travel cost;

End while;

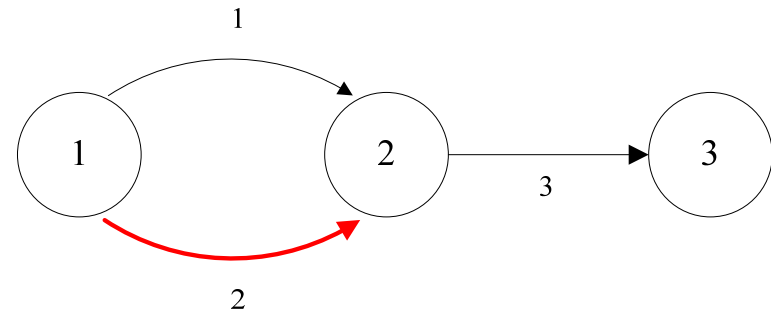
Numerical Experiments: Instance 1

Two-route network. The **red** arc is tolled.

Demand uncertainty set: **[7 13]**

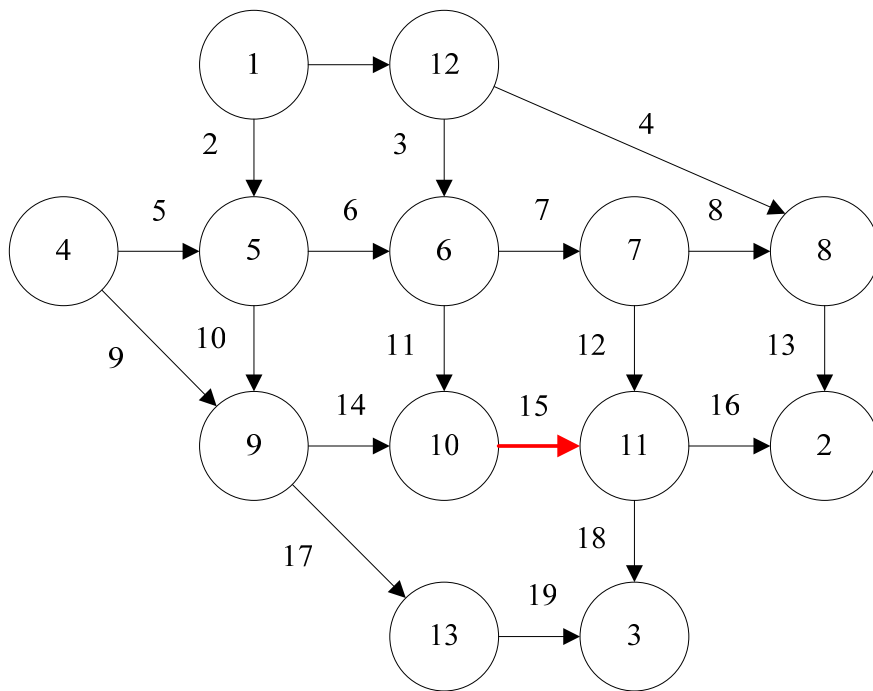
Uncertainty level: **0.3**

CPU time: **10941.8** seconds.



Optimal Toll		Simulation Result					
		Worst Case		Average		Std.	
		Val.	Imp. (%)	Val.	Imp. (%)	Val.	Imp. (%)
Robust Dynamic	2.0374	993.6	7.71	624.1	5.26	162.1	1.03
Robust Static	2.6998	1061.5	1.40	642.8	2.42	163.3	0.31
Nominal	5.0000	1076.6	-	658.7	-	163.8	-

Instance 2



25-route network. The **red** arc is tolled.
 Demand uncertainty set: **[50 150]**
 Uncertainty level: **0.5**
 CPU time: **62459.8** seconds

		Simulation Result					
		Worst		Average		Std.	
		Val.	Imp. (%)	Val.	Imp. (%)	Val.	Imp. (%)
Robust Dynamic	0.1534	79235.5	21.30	47010.6	2.52	8996.3	17.21
Nominal	0.0000	100695.9	-	48223.5	-	10865.8	-

Conclusion

- Robust optimization approach is applied to static and dynamic congestion pricing problems under demand uncertainty.
- In static case, we can derive a deterministic problem equivalent to the robust counterpart when we know the reaction function.
- For general cases, including robust DOTPEC, we propose BCPSO algorithm to find a robust solution.
- Numerical experiments show that the robust optimization approach leads to better solutions compared to the deterministic problem.
- Robust dynamic toll leads to better performance compared with robust static toll.
- In the future, the algorithms will be compared with alternative benchmark algorithms and will be tested for realistic and large scale network problems.

Thank you!

Questions and Comments?