

Multistage Air Traffic Flow Management Under Capacity Uncertainty: A Robust and Adaptive Optimization Approach

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Introduction

Weather Impact on Aviation Delays

Bertsimas Stock-Patterson Gupta Model (TFMP)

Model for Weather-Induced Capacity Uncertainty

Weather-front Based Approach

Polyhedral Characterization

Solution Methodologies

Robust Problem: Another Deterministic Instance

Adaptive Problem: Using Affine Policies

Computational Results

Conclusions

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Motivation: Impact of Weather

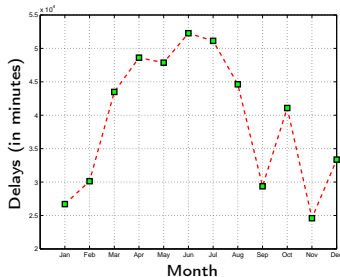


Figure: OPSNET¹ monthly delays for 2009.

- Increase in delays during the **summer months**.
- Thunderstorms account for **majority of air traffic delays** (60-70% monthly).

¹The Operations Network (OPSNET) is the official source of National Airspace (NAS) air traffic operations and delay data.

Capacity Uncertainty

- Lack of optimization models for **multi-airport/airspace settings**.
- **Large-scale nature** of the deterministic problem poses tractability challenge.

Capacity Uncertainty

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This Work

- **First application** of robust and adaptive optimization to Air Traffic Flow Management (ATFM) under capacity uncertainty.
- Aims to **bridge the gap** between deterministic network models and stochastic single-airport models.

Our Proposal

- Construct **low-dimensional** uncertainty sets.
- Propose **tractable solution methodologies** for the robust and adaptive problems.
- Approach in the spirit of **Robust Optimization (RO)** rather than **Stochastic Programming**.

Robust-Adaptive Paradigm

Robust Optimization

- Construct **uncertainty set** for possible data realizations
- Optimize **worst-case** objective (while ensuring feasibility for all scenarios)
- **Tractable** for a large class of optimization problems

Adaptive Optimization

- Paradigm for **multi-period** decision-making
- Decisions are adapted to capture the **progressive information** revealed over time
- Subclass: policy-based adaptability. **Affine policies** widely used and computationally tractable

Starting Point

The decision variables are:

$$w_{j,t}^f = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$s_{f,f'} = \begin{cases} 1, & \text{if there is a reversal b/w } f \text{ and } f', \\ 0, & \text{otherwise.} \end{cases}$$

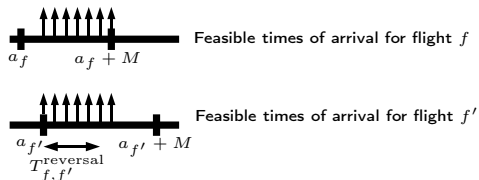


Figure: A reversible pair of flights $(f, f') \in \mathcal{R}$.

Starting Point

Concise description of the deterministic model (TFMP) ²:

$$\begin{aligned} IZ_{\text{TFMP}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}, \\ & \mathbf{w} \in \{0, 1\}^n. \end{aligned} \tag{1}$$

- TFMP: Superior integrality properties and computational efficiency.
- We introduce a uncertainty set \mathcal{U} on the capacity \mathbf{b} .

²Please refer to the end for complete description of the constraints. References: 1) Bertsimas and Stock-Patterson, *Operations Research*, 1998; and 2) Bertsimas and Gupta, *Operations Research*, 2011.

Robust and Adaptive Problems

Multi-stage Adaptive problem (Π_{Adapt}^T).

$$IZ_{\text{Adapt}} = \min_{\mathbf{w}_1, \mathbf{w}_i(\mathbf{b}_{[i]})} \left[\mathbf{c}'_1 \mathbf{w}_1 + \max_{\mathbf{b} \in \mathcal{U}} \left[\mathbf{c}'_2 \mathbf{w}_2(\mathbf{b}_{[2]}) + \cdots + \right. \right. \\ \left. \left. \max_{\mathbf{b} \in \mathcal{U}} \left[\mathbf{c}'_{T-1} \mathbf{w}_{T-1}(\mathbf{b}_{[T-1]}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_T \mathbf{w}_T(\mathbf{b}_{[T]}) \right] \right] \right]$$
$$\text{s.t. } \mathbf{A}_1 \mathbf{w}_1 + \sum_{i=2}^T \mathbf{A}_i \mathbf{w}_i(\mathbf{b}_{[i]}) \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \quad (\Pi_{\text{Adapt}}^T)$$
$$\mathbf{w}_i(\mathbf{b}_{[i]}) \in \{0, 1\}^{n_i}.$$

Robust problem (Π_{Rob}).

$$IZ_{\text{Rob}} = \min_{\mathbf{w}} \sum_{i=1}^T \mathbf{c}_i \mathbf{w}_i$$
$$\text{s.t. } \sum_{i=1}^T \mathbf{A}_i \mathbf{w}_i \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \quad (\Pi_{\text{Rob}})$$
$$\mathbf{w}_i \in \{0, 1\}^{n_i}.$$

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Weather-front based approach

Premise:

- Correlations over space and time
- Small number of weather-fronts moving through a day

Uncertain parameters:

- Time of arrival: $T_a \in \{\underline{T}_a, \dots, \bar{T}_a\}$.
- Duration: $d \in \{\underline{d}, \dots, \bar{d}\}$.
- Reduction in capacity: $\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}$.



Figure: Depiction of the capacity profile under a *weather-front* based uncertainty set for a single affected airspace element.

Motivation

- Practical evidence of the applicability of fitting step functions to actual capacity profiles ³
- Our proposal is compatible with existing available data

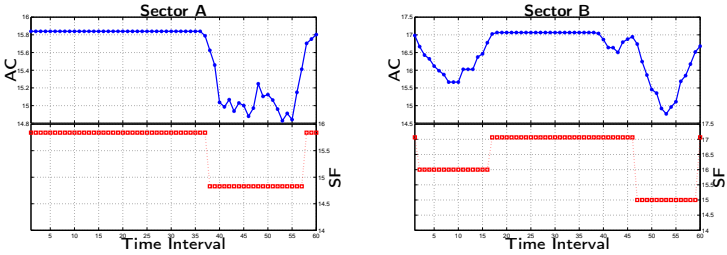


Figure: Applicability of step functions to model capacity profiles. **AC**: actual sector capacity; **SF**: step function capacity.

³capacity profiles for two sectors from data obtained from Lincoln Labs

Model for the airspace

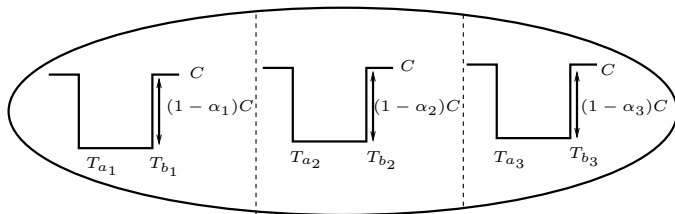


Figure: Traversal of a *weather-front* across the NAS. It has three phases over the course of its existence.

- Each affected airspace element associated with a 3-tuple (T_a, d, α) .

Weather-front: Discrete Uncertainty Set

Variables

$$y_t = \begin{cases} 1, & \text{if capacity drops by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$z_t = \begin{cases} 1, & \text{if capacity revives by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

$$b_t = \underbrace{C(1 - y_t)}_{\text{before drop}} + \underbrace{\alpha C y_t}_{\text{before revival}} + \underbrace{(1 - \alpha) C z_t}_{\text{after revival}}$$

after drop

Weather-front: Discrete Uncertainty Set

Description for a particular realization of α :

$$\mathcal{U}_\alpha = \{\mathbf{b} \in \mathbb{Z}_+^m \mid b_t = C(1 - y_t) + \alpha C y_t + (1 - \alpha) C z_t, \forall t \in \{\underline{T}_a, \dots, \overline{T}_b\};$$

$$b_t = C, \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \dots, \overline{T}_b\};$$

$$y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t;$$

$$y_{\overline{T}_a} = 1; z_{\underline{T}_b-1} = 0; z_{\overline{T}_b} = 1; y_t, z_t \in \{0, 1\}$$

Overall uncertainty set for the capacity profile:

$$\mathcal{U} = \left(\bigcup_{\alpha \in \{\underline{\alpha}, \dots, \overline{\alpha}\}} \mathcal{U}_\alpha \right)$$

Weather-front: Polyhedral Description

- The optimization problem under \mathcal{U} (discrete set) is equivalent to optimization problem under $\text{conv}(\mathcal{U})$.
- **Challenge:** A polyhedral description of $\text{conv}(\mathcal{U})$?

A polyhedral description of $\text{conv}(\mathcal{U}_\alpha)$.

$$\begin{aligned}\mathcal{P}_\alpha = \{ & \mathbf{b} \in \mathbb{R}_+^m \mid b_t = C(1 - y_t) + \alpha C y_t + (1 - \alpha) C z_t, \forall t \in \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & b_t = C, \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t; \\ & y_{\overline{T}_a} = 1; z_{\underline{T}_b - 1} = 0; z_{\overline{T}_b} = 1; 0 \leq y_t, z_t \leq 1\}\end{aligned}$$

Weather-front: Polyhedral Description

Theorem. \mathcal{P}_α is integral and is exactly the convex hull of \mathcal{U}_α .

$$\mathcal{U}_\alpha \subseteq \text{conv}(\mathcal{U}_\alpha) = \mathcal{P}_\alpha$$

Proof. Uses the technique of randomized rounding.

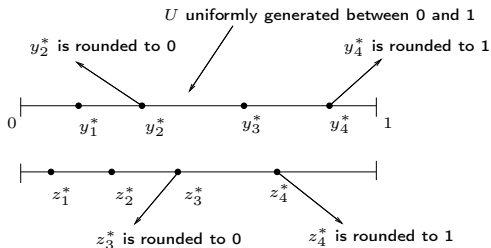


Figure: Illustration of the geometry of the *randomized rounding* algorithm for proving the integrality of polyhedron \mathcal{P}_α .

Theorem. $\mathcal{P}_{\underline{\alpha}}$ is exactly $\text{conv}(\mathcal{U})$, i.e.,

$$\text{conv}(\mathcal{U}) = \mathcal{P}_{\underline{\alpha}}$$

- \implies uncertainty governing α not required (input needed is only $\underline{\alpha}$).
- $(T_a, d, \alpha) \longrightarrow (T_a, d)$ for each airspace element.
- **Comparison with Hypercube Uncertainty Set.** 3 WFs (3 phases; 3 affected airspace elements, $|\overline{T}_a - \underline{T}_a + 1| = 3$, $|\overline{d} - \underline{d} + 1| = 3$ and $d \geq 3$). Hypercube: 162 parameters, WF: 54!

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Solving the Robust Problem

Definition. $\mathbf{b}_{\min} = (\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m)$, where $\underline{b}_i = \min \{b_i \mid \mathbf{b} \in \mathcal{U}\}$.

Theorem. For an arbitrary uncertainty set \mathcal{U} , the robust TFMP problem is equivalent to solving the following modified TFMP instance:

$$\begin{aligned} IZ_{\text{TFMPRob}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}_{\min}, \\ & \mathbf{w} \in \{0, 1\}^n. \end{aligned} \tag{2}$$

Solving the Robust Problem

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- Solving robust instance is equivalent to solving another TFMP (deterministic) instance (\implies retains the attractive integrality properties and computational efficiency of TFMP model).

Intuition for solving the Multi-stage Adaptive Problem.

Procedure.

- Construct a **polyhedral description** of the uncertainty set
- Divide the uncertainty set into **multiple simplices**
- An **affine policy** is optimal for each piece

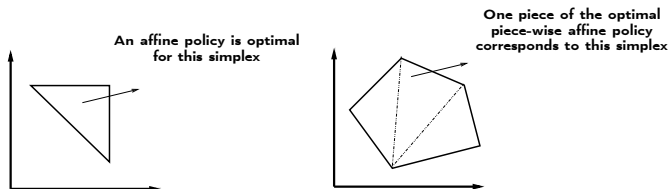


Figure: **Left:** A simplex uncertainty set in \mathbb{R}^2 . **Right:** A polyhedral set in \mathbb{R}^2 broken into three simplices.

Weather-front: # of Extreme Points

Exact Expression.

$$\mathcal{E} = \prod_{i=1}^k \prod_{j=1}^{P_i} \left((\bar{T}_{a_i} - \underline{T}_{a_i} + 1) \times (\bar{d}_i - \underline{d}_i + 1) \right)^{|\mathcal{W}_i^j(\mathcal{S})| + |\mathcal{W}_i^j(\mathcal{K})|}$$

Theorem.

Let k be the number of weather fronts and P be the maximum number of phases across all fronts. Let,

$$\tau = \max_i \{ \bar{T}_{a_i} - \underline{T}_{a_i} + 1, \bar{d}_i - \underline{d}_i + 1 \} \quad \Delta = \max_{i,j} \{ |\mathcal{W}_i^j(\mathcal{S})|, |\mathcal{W}_i^j(\mathcal{K})| \}$$

Then, \mathcal{E} can be upper bounded as follows:

$$\mathcal{E} \leq \tau^{4k*\Delta*P}$$

Weather-front: # of Extreme Points

τ	Δ	Upper Bound on \mathcal{E} ($k = 1$)	
		$P = 1$	$P = 2$
1	1	1	1
2	2	256	65536
2	3	4096	16777216
3	2	6561	43046721
3	3	531441	$2.8 \cdot 10^{11}$

Table: The bold numbers indicate the attractive cases from the point of view of obtaining a single affine policy.

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Performance of Robust Counterpart

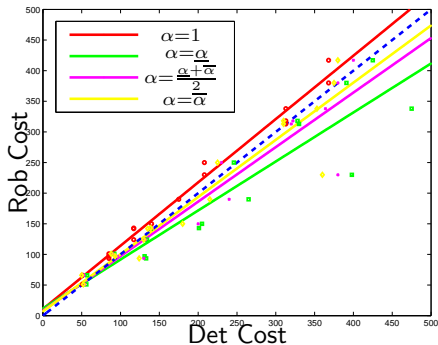


Figure: *Utility of Robust Solutions.*

- the robust schedules come at a small cost
- the cost of the protracted deterministic solution is generally higher

Performance of Robust Counterpart

Region (# of Flights)	Z_{Det}	Sol. Time (sec.)	Z_{Rob}	Sol. Time (sec.)	% Nonint	Uncertainty Set (# of EPs)
<i>North-East</i> (500-1000)	208	378	229	407	0	81
	227	515	230	643	0	256
	145	636	161.5	638	0.15	729
	281	466	281	533	0	14677618
	935	4839	971.5	4721	0	20736000
<i>Central</i> (500-1000)	650	1996	677	1984	0	81
	193	103	193	95	0	256
	647	1770	675	1779	0	729
	935	4685	1013	4607	0	4299816
	240	401	248	402	0.10	1679616

Table: Computational Experience with TWURob.

Performance of Adaptive Counterpart

Region (# of Flights)	Z_{Rob}	Sol. Time (sec.)	Z_{Adapt}	Sol. Time (sec.)	% Nonint	Uncertainty Set (# of EPs)
<i>North-East</i> (100-150)	92	84	92	265	0	81
	29.5	73	29.5	491	0	128
	54	55	54	369	0	256
	113	127	112	326	0	4096
<i>Central</i> (100-150)	27	43	27	813	0	1696
	22	34	22	287	0	225
	86	98	86	257	0	729
	35	63	35	382	0	1728

Table: Computational Experience with TWUAdapt.

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Summary

- An effort to address **stochasticity in a network ATFM setting**.
- **Low-dimensional**, realistic model of uncertainty set.
- **Tractable solution methodologies** for the robust and adaptive problem.

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- **Low-dimensional**, realistic model of uncertainty set.
- **Tractable solution methodologies** for the robust and adaptive problem.

Takeaways

- Robust problem equivalent to solving another **deterministic instance**.
- **Affine policies are optimal** for the LP relaxation of the adaptive equivalent.

Questions?

Thank You!

Air Traffic Flow Management (ATFM)

What is ATFM?

- Set of **strategic processes** that reduce congestion costs.
- **Ground Delay Programs (GDPs)** - one of the major ATFM tools.
- **Airspace Flow Programs (AFPs)** - a more recently used tool; others include assigning air-borne delays, dynamic re-routing and speed control.

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Element of Stochasticity

- Only handled reasonably in the context of GDPs.
- Multi-airport/Airspace settings not really addressed.

Two Paradigms

- **Lagrangian**: trajectory-based.
- **Eulerian**: control of aggregate flow.

Goal

- Minimize the **total delay costs** across all flights.
- Focus is on maximizing **system throughput**.

Limitations

- **Deterministic models**.

Papers

- **Ball, Hoffman, Odoni and Rifkin (2003)** - A Stochastic IP with Dual Network Structure.
- **Kotynek, Richetta (2006)** - Relating Equity and Integrality.
- **Mukherjee, Hansen (2007)** - A Dynamic Stochastic Model.

Starting Point: Bertsimas Stock-Patterson Model (1998)

Decision Variables

$$w_{j,t}^f = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

- Definition with “by” rather than “at” critical for models’ excellent performance.

Objective Function

$$IZ_{\text{TFMP}} = \min_{\mathbf{w}, \mathbf{s}} \sum_{f \in \mathcal{F}} \left(\delta \cdot \left(\sum_{t \in T_{\text{dest}}^f} t \cdot (w_{\text{dest}_f, t}^f - w_{\text{dest}_f, t-1}^f) - a_f \right) + (1 - \delta) \right. \\ \left. \left(\sum_{t \in T_{\text{orig}}^f} t \cdot (w_{\text{orig}_f, t}^f - w_{\text{orig}_f, t-1}^f) - d_f \right) \right) + \lambda \cdot \left(\sum_{(f, f') \in \mathcal{R}} s_{f, f'} \right)$$

Capacity Constraints

$$\sum_{f \in \mathcal{F}: \text{orig}_f = k} (w_{k,t}^f - w_{k,t-1}^f) \leq D_k(t), \quad \forall k \in \mathcal{K}, t \in \mathcal{T}, \quad (3)$$

$$\sum_{f \in \mathcal{F}: \text{dest}_f = k} (w_{k,t}^f - w_{k,t-1}^f) \leq A_k(t), \quad \forall k \in \mathcal{K}, t \in \mathcal{T}, \quad (4)$$

$$\sum_{f \in \mathcal{F}: j \in \mathcal{S}_f, j' = L_j^f} (w_{j,t}^f - w_{j',t}^f) \leq S_j(t), \quad \forall j \in \mathcal{S}, t \in \mathcal{T}, \quad (5)$$

- Constraints satisfying the departure, arrival and sector capacities respectively.

Connectivity Constraints

$$w_{j,t}^f - w_{j',t-l_{fj'}}^f \leq 0, \quad \forall f \in \mathcal{F}, \quad t \in T_j^f, \quad j \in \mathcal{S}^f : j' = P_j^f, \quad (6)$$

$$w_{\text{orig}_f,t}^f - w_{\text{dest}_{f'},t-s_f}^{f'} \leq 0, \quad \forall (f, f') \in \mathcal{C}, \quad \forall t \in T_k^f, \quad (7)$$

$$w_{j,t-1}^f - w_{j,t}^f \leq 0, \quad \forall f \in \mathcal{F}, \quad j \in \mathcal{S}^f, \quad t \in T_j^f, \quad (8)$$

- Constraints satisfying sector, flight and time connectivity respectively.
- Connectivity constraints are facet-defining which is why this model performs very well computationally.

Fairness Constraints

$$w_{\text{dest}_{f'},t}^{f'} \leq w_{\text{dest}_f,t}^f + s_{f,f'}, \quad \forall (f, f') \in \mathcal{R}, \quad t \in T_{f,f'}^{\text{reversal}}, \quad (9)$$

$$w_{\text{dest}_f,t}^f \leq w_{\text{dest}_{f'},t}^{f'} + 1 - s_{f,f'}, \quad \forall (f, f') \in \mathcal{R}, \quad t \in T_{f,f'}^{\text{reversal}}, \quad (10)$$

- Constraints model a reversal.
- Fairness corresponds to controlling the number of reversals.

Data and Setup

- Airspace: 10 by 10 grid (100 sectors). 55 major US airports mapped to one of these 100 sectors.
- Different regions of the airspace subjected to simulated weather-fronts. (T_a, d, α) generated randomly from appropriate intervals.
- Solution Tool: *Robust Optimization Made Easy* (ROME).

Characteristics of Robust Solutions

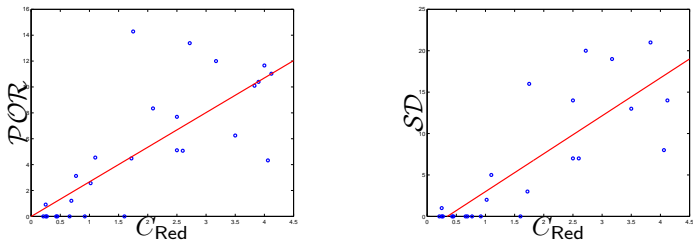


Figure: **Left:** Price of robustness (POR) as a function of capacity reduction. **Right:** Schedule deviation (SD) as a function of capacity reduction.

- Approximately **linear relationship** of POR and SD with C_{Red} (the red line corresponds to the best linear fit.)

Relating the Robust and Adaptive problems

Proposition

Consider the multi-stage adaptive optimization problem Π_{Adapt}^T and its robust counterpart Π_{Rob} . If $\mathbf{b}_{\min} \in \mathcal{U}$, then,

$$IZ_{\text{Adapt}} = IZ_{\text{Rob}}.$$

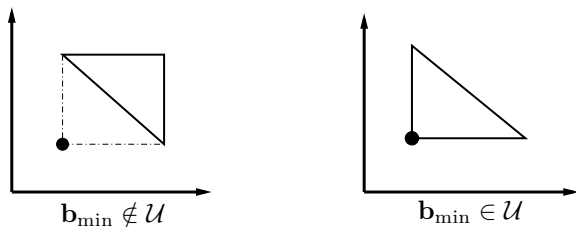


Figure: Example uncertainty sets with and without \mathbf{b}_{\min} (black filled circle denotes \mathbf{b}_{\min}).

Toolbox to solve the Adaptive Problem

Theorem

Consider the problem $\Pi_{\text{Adapt}}^T(\mathcal{U})$ such that \mathcal{U} is a simplex. Then, there is an optimal multi-stage solution $\hat{\mathbf{w}}_i(\mathbf{b})$ such that $\hat{\mathbf{w}}_i(\mathbf{b})$ are affine functions of \mathbf{b} , i.e., for all $\mathbf{b} \in \mathcal{U}$,

$$\hat{\mathbf{w}}_i(\mathbf{b}) = \mathbf{P}_i \mathbf{b} + \mathbf{q}_i, \quad (11)$$

where $\mathbf{P}_i \in \mathbb{R}^{n_i \times m}$, $\mathbf{q}_i \in \mathbb{R}^{n_i}$.

Weather-front: # of Extreme Points

Description (Denoted By)	Uncertainty Set	# of Extreme Points
1 WF, Single airspace element, one phase (EP_i)	$\mathcal{P}_{\underline{\alpha}_i} = \{ \mathbf{b} \in \mathbb{R}_+^m \mid b_t = C(1-y_t) + \underline{\alpha}_i C y_t \\ + (1-\underline{\alpha}_i) C z_t, \forall t \in \{ \underline{T}_{a_i}, \dots, \overline{T}_{b_i} \}; \\ b_t = C, \forall t \in \mathcal{T} \setminus \{ \underline{T}_{a_i}, \dots, \overline{T}_{b_i} \}; \\ y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t; 0 \leq y_t, z_t \leq 1 \}$	$(\overline{T}_{a_i} - \underline{T}_{a_i} + 1) \times (\overline{d}_i - \underline{d}_i + 1)$
1 WF, Airspace, one phase (\mathcal{E}_i^j)	$\mathcal{P}_{\underline{\alpha}_i}^{AS, \mathcal{OP}} = \bigoplus_{k \in \mathcal{A}} \mathcal{P}_{\underline{\alpha}_i}^k$	$EP_i^{ \mathcal{W}_i^j(S) + \mathcal{W}_i^j(\mathcal{K}) }$
1 WF, Airspace, all phases (\mathcal{E}_i)	$\mathcal{P}_{\underline{\alpha}_i}^{AS, \mathcal{AP}} = \bigoplus_{j=1}^{p_i} \mathcal{P}_{\underline{\alpha}_i}^j$	$\prod_{j=1}^{p_i} \mathcal{E}_i^j$
k WF, Airspace, all phases (\mathcal{E})	$\mathcal{P}_{\underline{\alpha}}^{AS, \mathcal{AP}} = \bigoplus_{i=1}^k \mathcal{P}_{\underline{\alpha}_i}$	$\prod_{i=1}^k \mathcal{E}_i$

Table: Number of extreme points for the weather-front based polytope.
 AS denotes airspace, \mathcal{OP} denotes one-phase and \mathcal{AP} denotes all-phases.

Optimization Under Discrete Uncertainty Set

OptiDU denotes optimization problem under discrete uncertainty set (\mathcal{U}_0 : discrete uncertainty set).

$$\begin{aligned} Z_{\text{OptiDU}} = \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}_0. \end{aligned} \tag{12}$$

- Work with $\text{conv}(\mathcal{U}_0)$ (convex hull of \mathcal{U}_0) to enable a polyhedral description.
- The optimization problem under \mathcal{U}_0 (discrete set) is equivalent to optimization problem under $\text{conv}(\mathcal{U}_0)$.

Notation.

The model's formulation requires definition of the following notation:

$\mathcal{K} \equiv$ set of airports,

$\mathcal{W} \equiv$ set of airlines,

$\mathcal{W}^k \equiv$ set of flights belonging to airline k ,

$\mathcal{S} \equiv$ set of sectors,

$\mathcal{S}^f \subseteq \mathcal{S} \equiv$ set of sectors that can be flown by flight f ,

$\mathcal{F} \equiv$ set of flights,

$\mathcal{T} \equiv$ set of time periods,

$\mathcal{C} \equiv$ set of pairs of flights that are continued,

$\mathcal{R} \equiv$ set of pairs of flights that are reversible, (see definition below)

$\mathcal{P}_i^f \equiv$ set of sector i 's preceding sectors,

$\mathcal{L}_i^f \equiv$ set of sector i 's subsequent sectors,

Notation (Continued).

$D_k(t) \equiv$ departure capacity of airport k at time t , (13)

$A_k(t) \equiv$ arrival capacity of airport k at time t ,

$S_j(t) \equiv$ capacity of sector j at time t ,

$d_f \equiv$ scheduled departure time of flight f ,

$a_f \equiv$ scheduled arrival time of flight f ,

$s_f \equiv$ turnaround time of an airplane after flight f ,

$orig_f \equiv$ airport of departure of flight f ,

$dest_f \equiv$ airport of arrival of flight f ,

$l_{fj} \equiv$ minimum number of time units that flight f must spend in sector j ,

$D \equiv$ maximum permissible delay for a flight,

$T_j^f = [\underline{T}_j^f, \overline{T}_j^f] \equiv$ set of feasible time periods for flight f to arrive in sector j ,

$\underline{T}_j^f \equiv$ first time period in the set T_j^f ,

$\overline{T}_j^f \equiv$ last time period in the set T_j^f .