Tailored Supply Chain Decision-Making Under Price-Sensitive Stochastic Demand and Delivery Uncertainty

Saibal Ray    Shanling Li    Yuyue Song

Faculty of Management, McGill University, Montreal, Quebec, Canada

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Appendix B (Proofs for Lemmas and Theorems)

Derivations of $m$ and $v$ and proof of Lemma 1

\[ E(\xi) = m(\rho_0) = \int_0^{+\infty} x d\Phi(\frac{x + S - \mu L_0}{\sigma L_0}) = \int_{\rho_0}^{+\infty} \phi(y)(\sigma L_0 y + \mu L_0 - S)dy \]

\[ = \sigma L_0 \phi(\rho_0) + (\mu L_0 - S)[1 - \Phi(\rho_0)] = \sigma L_0 \{\phi(\rho_0) - \rho_0[1 - \Phi(\rho_0)]\}. \]

As \( \int_{\rho_0}^{+\infty} y^2 \phi(y)dy = \rho_0 \phi(\rho_0) + 1 - \Phi(\rho_0) \), we get

\[ E(\xi^2) = \int_0^{+\infty} x^2 d\Phi(\frac{x + S - \mu L_0}{\sigma L_0}) = \int_{\rho_0}^{+\infty} \phi(y)(\sigma L_0 y + \mu L_0 - S)^2dy \]

\[ = \sigma^2 L_0 \{\rho_0^2[1 - \Phi(\rho_0)] - 2\rho_0 \phi(\rho_0) + \int_{\rho_0}^{+\infty} y^2 \phi(y)dy \} = \sigma^2 L_0 \{(\rho_0^2 + 1)[1 - \Phi(\rho_0)] - \rho_0 \phi(\rho_0)\}. \]

Hence, \( v(\rho_0) = E(\xi^2) - (E(\xi))^2 = \sigma^2 L_0 \{(\rho_0^2 + 1)[1 - \Phi(\rho_0)] - \rho_0 \phi(\rho_0)\} - m(\rho_0)^2. \)

Note that \( \phi'(\rho_0) = -\rho_0 \phi(\rho_0) \). By taking derivative of $m$ with respect to (wrt) \( \rho_0 \), \( m'(\rho_0) = \sigma L_0 [-1 + \Phi(\rho_0)] \leq 0 \), and \( m''(\rho_0) = \sigma L_0 \phi(\rho_0) \geq 0 \). Hence \( m(\rho_0) \) is decreasing and convex for \( \rho_0 \in (-\infty, +\infty) \). Taking derivative of \( v \) wrt \( \rho_0 \), we get \( v^{(\rho_0)}(\rho_0) = 2\Phi(\rho_0)[\rho_0(1 - \Phi(\rho_0)) - \phi(\rho_0)] \) and \( \frac{v''(\rho_0)}{\sigma L_0} = 2(1 - \Phi(\rho_0))\Phi(\rho_0) - 2\{\phi(\rho_0) - \rho_0(1 - \Phi(\rho_0))\} \phi(\rho_0) \). As \( [\phi + (\Phi - 1)\rho_0]_{\rho_0=0} = \frac{1}{\sqrt{\pi}} \lim_{\rho_0 \to +\infty}[\phi + (\Phi - 1)\rho_0] = 0 \), and \( (\phi + (\Phi - 1))' = (\Phi - 1) < 0 \), we get \( 2[\phi + (\rho_0(\Phi - 1)) > 0 \), which implies \( \rho_0(1 - \Phi(\rho_0)) - \phi(\rho_0) < 0 \). Hence, \( v(\rho_0) \) is decreasing for \( \rho_0 \in (-\infty, +\infty) \). Since \( \rho_0 \) is increasing in \( p \) and \( S \), \( m \) and \( v \) are decreasing in those variables. Similarly the effects of \( L_0 \) and \( \sigma_d \) can also be proved.\)

Proof of Theorem 1

First we provide some results, which will be used in our subsequent analysis.

Lemma 4 a) \( \rho_0 \phi(1 - \Phi) - \phi^2 + (1 - \Phi)^2 > 0 \) for \( \rho_0 \in (-\infty, +\infty) \); b) \( 2v[v''\phi - v'\phi] - v^2\phi > 0 \) for \( \rho_0 \in (-\infty, +\infty) \); c) There exists a positive constant \( \alpha > 0 \) such that \( v'\phi + v''(1 - \Phi) < 0 \) for \( \rho_0 \in (-\infty, \alpha) \) and \( v'\phi + v''(1 - \Phi) > 0 \) for \( \rho_0 \in (\alpha, +\infty) \).

Proof. a) As \( [\rho_0 \phi(1 - \Phi) - \phi^2 + (1 - \Phi)^2]_{\rho_0=0} = \frac{\pi - 2}{4\pi} \) and \( \lim_{\rho_0 \to +\infty}[\rho_0 \phi(1 - \Phi) - \phi^2 + (1 - \Phi)^2] = 0 \), it is sufficient to show that \( \rho_0 \phi(1 - \Phi) - \phi^2 + (1 - \Phi)^2 \) is decreasing, i.e., \( (\rho_0 \phi(1 - \Phi) - \phi^2 + (1 - \Phi)^2)' = \phi((\Phi - 1)(\rho_0^2 + 1) + \rho_0 \phi) < 0 \). This is equivalent to proving \( (\Phi - 1)(\rho_0^2 + 1) + \rho_0 \phi < 0 \). As the value of the left hand of the last inequality at \( \rho_0 = 0 \) is \( -\frac{1}{2} \) and the limit as \( \rho_0 \to +\infty \) is 0, it is sufficient to show that \( (\Phi - 1)(\rho_0^2 + 1) + \rho_0 \phi \) is increasing, i.e., \( ((\Phi - 1)(\rho_0^2 + 1) + \rho_0 \phi)' = 2[\phi + (\Phi - 1)\rho_0] > 0 \), which we already showed to be true in the proof of Lemma 1.
b) From the expressions of \( v'(\rho) \) and \( v''(\rho_0) \) it is clear that the original inequality is equivalent to \( 4v\Phi(1-\Phi) - \frac{v'^2}{\sigma_{\rho_0}^2} > 0 \). By substituting the expressions of \( v \) and \( v' \) into this inequality, we basically need to prove \((1-\Phi)[(\rho_0^2+1)(1-\Phi)-\rho_0\phi-\frac{m(\rho_0)^2}{\sigma_{\rho_0}^2}] > \Phi[\rho_0^2(1-\Phi)^2 + \phi^2 - 2\rho_0\phi(1-\Phi)]\), i.e., \( \rho_0\phi(1-\Phi) - \phi^2 + (1-\Phi)^2 > 0 \) and this is true by part (a).

c) By substituting the expressions of \( v' \) and \( v'' \), we only need to study the sign of the following quantity, \( \Phi(1-\Phi)^2 - \phi[\phi - \rho_0(1-\Phi)] \) (say, \( f(\rho_0) \)). Differentiating \( f \) and simplifying we can show that

\[
 f''(\rho_0) = 2\rho_0\phi \frac{\phi - \rho_0(1 - \Phi)}{1 - \Phi}.
\]

By part (a), \( \phi[\phi - \rho_0(1 - \Phi)] - (1 - \Phi)^2 < 0 \), \( \rho_0 \in (-\infty, +\infty) \). Hence, we have \( f''(\rho_0) < 0 \), \( \rho_0 \in (0, +\infty) \) and \( f''(\rho_0) > 0 \), \( \rho_0 \in (-\infty, 0) \). Also \( f(0) < 0, \lim_{\rho_0 \to -\infty} f(\rho_0) = 0 \). Hence, proved.

Now we can proceed with the proof of Theorem 1. Let the uncertainty-induced cost portion of the profit function be represented by \( L(\rho_0, p) = K\sigma(\rho_0, p) + \sigma_{L_0}h_0[\rho_0\Phi(\rho_0) + \phi(\rho_0)] \). First, for a fixed \( p \), we prove the unimodality of \(-L\). Note that \( \sigma(\rho_0, p) = \sqrt{c_1\lambda(p)^2 + c_2 + v(\rho_0)} \).

\[
\frac{\partial\sigma}{\partial\rho_0} = \frac{1}{2}\frac{v'}{\sigma} \quad \frac{\partial^2\sigma}{\partial\rho_0^2} = \frac{12v''\sigma^2 - v'^2}{4\sigma^3}, \quad \text{and} \quad \frac{\partial^2\sigma}{\partial p\partial\rho_0} = -\frac{c_1}{2} \frac{v'\lambda\lambda'}{\sigma^3}.
\]

Differentiating \( L \) with respect to \( \rho_0 \) and using Lemma 4, we get

\[
K^{-1} \frac{\partial^2 L(\rho_0, p)}{\partial\rho_0^2} |_{(\frac{\partial\rho_0(p, p)}{\partial\rho_0}) = 0} = \left\{ \frac{\partial^2\sigma}{\partial\rho_0^2} + \frac{\sigma_{L_0}h_0}{K} \phi(\rho_0) \right\} = \left( \frac{\partial^2\sigma}{\partial\rho_0^2} - \frac{v'\phi}{2\sigma\Phi} \right) \geq \frac{(2v'' - v'^2)\Phi - 2v'\phi v}{4\Phi\sigma^3} > 0.
\]

Hence, \(-L\) is unimodal in \( \rho_0 \) for a given \( p \). The optimal \( \rho_0, \rho_0(p) \), then must satisfy \( K \frac{\partial\sigma(\rho_0, p)}{\partial\rho_0} + \sigma_{L_0}h_0\Phi(\rho_0) = 0 \). Taking derivative with respect to \( p \) on both sides and simplifying, we have

\[
\frac{dp_0(p)}{dp} = \left( -\frac{\partial^2\sigma}{\partial p\partial\rho_0} \right) \left\{ \frac{\partial^2\sigma}{\partial\rho_0^2} + \frac{\sigma_{L_0}h_0}{K} \phi(\rho_0) \right\}^{-1} > 0.
\]

From the above proof and Lemma 4, we see

\[
\rho_0(p)' = \frac{2c_1\lambda\lambda'v'\Phi}{\Phi[2\sigma^2v'' - v'^2] - 2\phi v'\sigma^2} \leq \frac{2c_1\lambda\lambda'v'\Phi}{(c_2 + c_1\lambda^2)[2\phi v'' - 2\phi v']} \leq \frac{2c_1\lambda\lambda'v}{v'(c_2 + c_1\lambda^2)}.
\]

From \( \frac{d^2\Pi(\rho_0(p))}{dp^2} = 0 \), we get \((h + b)\phi(\mu^C) = \frac{[\lambda + (p-c)\lambda']\sigma}{c_1\lambda\lambda'}\). Therefore,

\[
\frac{d^2\Pi(\rho_0(p))}{dp^2} |_{(\frac{d\Pi(\rho_0(p))}{dp}) = 0} = 2\lambda' + (p-c)\lambda'' - \frac{[\lambda + (p-c)\lambda']\sigma}{c_1\lambda\lambda'} \{c_1 (\lambda'^2 + \lambda\lambda')\sigma^2 - c_1 (\lambda\lambda')^2 \} - \frac{c_1\lambda\lambda'v'}{2\sigma^3}\rho_0(p)' \leq 2\lambda' + (p-c)\lambda'' - \frac{[\lambda + (p-c)\lambda']\sigma}{\sigma^2\lambda\lambda'} \{c_1 (\lambda'\sigma^2) \} = 2\lambda' - \frac{\lambda\lambda''}{\lambda'} \leq 0.
\]
The last inequality is valid because $\lambda$ is decreasing and concave.\(\diamondsuit\)

**Proof of Theorem 2**

If the NLTD of the retailer is a constant, i.e., $\text{Var}(L) = 0$, $\sigma$ is independent of $p$ and all the results in the theorem hold true (in fact under those conditions $\pi(p)$ is concave in $p$). Now, suppose $\text{Var}(L) > 0$. Differentiating $\pi(p)$ with respect to $p$ and simplifying, we get

$$
\frac{d^2\pi(p)}{dp^2} \bigg|_{(p)''=0} = [(p-w)\lambda(p)]'' - \frac{\lambda(p)^2 + \lambda(p)\lambda(p)''}{\lambda(p)\lambda(p)'} - \frac{c_1\lambda(p)\lambda(p)'}{\sigma^2} \\
\text{[(p-w)\lambda(p)']].}
$$

Note that $-\frac{c_1\lambda(p)\lambda(p)'}{\sigma^2} < -\frac{\lambda(p)'}{\lambda(p)}$. Also, $[(p-w)\lambda(p)']' < 0$ from $\pi(p)' = 0$. Hence, for unimodality (i.e., $\frac{d^2\pi(p)}{dp^2} \bigg|_{(p)''=0} < 0$) we only need to show that $[(p-w)\lambda(p)]'' - \frac{\lambda(p)''}{\lambda(p)} [(p-w)\lambda(p)'] < 0$. This is equivalent to $2\lambda(p)' - \frac{\lambda(p)}{\lambda(p)'} < 0$. This is true since $\lambda(p)$ is decreasing and concave.\(\diamondsuit\)

**Proof of Corollary 2**

We only show the derivation for the effect of $w$ on optimal $p$ and $R$, since others can be deduced similarly. Partial differentiation of (5) with respect to $w$ yields, $(\frac{\partial \rho}{\partial w})^{-1} = 1 + \frac{\lambda^2 - \lambda''}{\lambda^2} - \frac{Kc_1\lambda}{\sigma} (\sigma - \frac{c_1\lambda^2}{\sigma})$. Since $\lambda$ is decreasing concave, and $\sigma^2 \geq c_1\lambda^2$, $\frac{\partial \rho}{\partial w} > 0$. As $w$ increases, $p(w)$ increases, and so $\mu$ and $\sigma$ decreases. This implies that $R$ would decrease with $w$ (since $\rho^D$ is a constant).\(\diamondsuit\)

**Proof of Lemma 2**

For $\text{Var}(L) = 0$, a simple comparison of the first order conditions wrt $p$ and $\rho_0$ for the centralized and decentralized system can prove that $p^D > p^C$. For $\text{Var}(L) > 0$, $p^D$ must satisfy $\{[p + \frac{\lambda}{\lambda} - Kc_1\frac{\lambda}{\sigma(p_0^D, p)} - c\lambda]\}'_{p=p^D} = 0$, while $p^C$ satisfies $\{(p-c)\lambda - K\sigma(p_0^C, p)\}'_{p=p^C} = 0$. Now, if $\rho_0^D \geq \rho_0^C$, we get $\sigma(p_0^D, p) \leq \sigma(p_0^C, p)$. Using this fact, we have

$$\frac{\lambda^2}{\lambda} - Kc_1\frac{\lambda^2}{\sigma(p_0^D, p)} + K\sigma(p_0^C, p)' > -Kc_1\lambda\lambda'\frac{1}{\sigma(p_0^D, p)} - \frac{1}{\sigma(p_0^C, p)} + \frac{\sigma(p_0^D, p)^2 - c_1\lambda^2}{\sigma(p_0^D, p)^3} > 0.$$

Hence, if $\rho_0^D \geq \rho_0^C$, $p^D > p^C$. For $\rho_0^D < \rho_0^C$, let $X = \lambda(p)^2\text{Var}(L) + \sigma^2 E(L)$. Then,

$$Q_1 = \frac{1}{\sigma(p_0^C, p)} - \frac{1}{\sigma(p_0^D, p)} < \frac{1}{\sqrt{X}} - \frac{1}{\sqrt{X}} = \frac{1}{\sqrt{X}} - \frac{1}{\sqrt{X + \sigma^2 E(L)}} \leq \frac{v(\rho_0^D)}{2\sqrt{X + \sigma^2 E(L)}}.$$

In order to show $p^D > p^C$, it is sufficient to show $Q_1 \leq \frac{c_2 + v(\rho_0^D)}{(X + \sigma^2 E(L))^{\frac{1}{2}}}. Using the estimation of $Q_1$, it is sufficient to show $X \geq v(\rho_0^D)$. As $\lambda(p)^2\text{Var}(L) > 0$ it is sufficient to show $\sigma^2 E(L) \geq v(\rho_0 = -\infty) = \sigma^2 \rho_0$, i.e., $E(L) \geq L_0$. Hence, under that assumption, $p^D > p^C$ even for $\text{Var}(L) > 0$. On the other hand, based on the expression of $\pi_0(p_0^C, p)$, it is easy to show that $p_0^D \leq \rho_0.$\(\diamondsuit\)
Proofs of Theorem 3 and Corollary 3

First we prove the unimodality of \( \pi_0(\rho_0, p) \) for any fixed \( \rho_0 \in (-\infty, \bar{p}_0] \). Note that we are interested in \( p \in [p^C, P^u] \) (refer to Lemma 2). Let \( g(\rho_0, p) = \frac{\lambda^2}{\sigma} \). Then

\[
\frac{\partial \pi_0(\rho_0, p)}{\partial p} = \{(p + \frac{\lambda}{\lambda_1} - c)\lambda\}' - K_c \frac{\partial g}{\partial p} \quad \text{and} \quad \frac{\partial^2 \pi_0(\rho_0, p)}{\partial p^2} = \{(p + \frac{\lambda}{\lambda_1} - c)\lambda\}'' - K_c \frac{\partial^2 g}{\partial p^2}.
\]

(8)

Based on \( \frac{\partial g(\rho_0, p)}{\partial p} = 0 \), we get \( K_c = \{(p + \frac{\lambda}{\lambda_1} - c)'\} \left(\frac{\partial g}{\partial p}\right)^{-1} \). In order to study the unimodality, we only need to show \( \frac{\partial^2 \pi_0(\rho_0, p)}{\partial p^2} \mid_{\left(\frac{\partial g(\rho_0, p)}{\partial p} = 0\right)} < 0 \). Note that

\[
\frac{\partial g}{\partial p} = -\frac{\lambda^2 v'}{2\sigma^3}, \quad \frac{\partial^2 g}{\partial p^2} = -\frac{\lambda^2(2\sigma^2 v'' - 3v'^2)}{4\sigma^5}, \quad \frac{\partial^2 g}{\partial p^2} = \frac{\lambda^2}{\sigma^3}, \quad \frac{\partial^2 g}{\partial p^2} = \frac{\lambda^2}{\sigma^3} \frac{(2\sigma^2 - c_1 \lambda^2)(2\sigma^2 - c_1 \lambda^2)}{\sigma_3} + \frac{\lambda^2}{\sigma_3} \frac{(2\sigma^2 - c_1 \lambda^2)(2\sigma^2 - c_1 \lambda^2)}{\sigma_3}.
\]

We can now obtain the expressions for the derivatives of \( \pi_0(\rho_0, p) \). Also note that as \( \frac{\partial g}{\partial p} < 0 \), proving \( \frac{\partial^2 \pi_0(\rho_0, p)}{\partial p^2} \mid_{\left(\frac{\partial g(\rho_0, p)}{\partial p} = 0\right)} < 0 \) is equivalent to proving \( \Delta = \{\lambda(p + \frac{\lambda}{\lambda_1} - c)\}' \left(\frac{\partial g}{\partial p}\right)^{-1} - \{(p + \frac{\lambda}{\lambda_1} - c)\}' \left(\frac{\partial g}{\partial p}\right)^{-1} > 0 \). It is possible to show that (provided \( \lambda \lambda'' + \lambda \lambda''' \geq 0 \forall p \in [p^C, P^v] \))

\[
\Delta = \lambda(p + \frac{\lambda}{\lambda_1} - c)\lambda' \left(\frac{\partial g}{\partial p}\right)^{-1} - \{(p + \frac{\lambda}{\lambda_1} - c)\} \left(\frac{\partial g}{\partial p}\right)^{-1} > \lambda(p + \frac{\lambda}{\lambda_1} - c)\lambda' \left(\frac{\partial g}{\partial p}\right)^{-1} - \{(p + \frac{\lambda}{\lambda_1} - c)\} \left(\frac{\partial g}{\partial p}\right)^{-1}.
\]

As \( \{p + \frac{\lambda}{\lambda_1} - c\}' < 0 \) by \( \frac{\partial g(\rho_0, p)}{\partial p} = 0 \), if \( 3c_1 \lambda^2 \leq 2\sigma^2 \) for any \( \rho_0 \in (-\infty, \bar{p}_0] \) and \( p \in [p^C, P^u] \), we have \( \Delta > 0 \), which proves the unimodality of \( \pi_0(\rho_0, p) \) for any fixed \( \rho_0 \in (-\infty, \bar{p}_0] \). Substituting \( \sigma \) from (1), \( 3c_1 \lambda(p)^2 \leq 2\sigma^2 \Rightarrow \lambda(p)^2 \text{Var}(L) - 2(\sigma_d^2 E(L) + v(\rho_0)) \leq 0 \). Note that the left hand side of the last inequality is a decreasing function of \( \rho_0 \) and increasing function of \( \rho_0 \). Lemma 2 proves that if \( E(L) \geq L_0 \), then \( p^D > p^C \) and \( \rho_0^D \leq \bar{p}_0 \). Hence the largest value of the left hand side of the inequality is \( \lambda(p^C)^2 \text{Var}(L) - 2(\sigma_d^2 E(L) + v(\bar{p}_0)) \). So \( \lambda(p^C)^2 \text{Var}(L) - 2(\sigma_d^2 E(L) + v(\rho_0)) \leq 0 \) and \( E(L) \geq L_0 \) are sufficient for \( 3c_1 \lambda(p)^2 \leq 2\sigma^2 \), and therefore along with \( \lambda \lambda'' + \lambda \lambda''' \geq 0 \) for the unimodality of \( \pi_0(\rho_0, p) \) for any fixed \( \rho_0 \in (-\infty, \bar{p}_0] \), where \( p \in [p^C, P^u] \).

For any given \( \rho_0 \in (-\infty, \bar{p}_0] \), there then exists a unique \( p(\rho_0) \) which maximizes \( \pi_0(\rho_0, p) \).

Note that if \( p(\rho_0) \in (p^C, P^u) \), it satisfies \( K_c \frac{\partial g}{\partial p} = \{(p + \frac{\lambda}{\lambda_1} - c)\}' \). Taking derivative with respect to \( \rho_0 \) on both sides we get \( (p + \frac{\lambda}{\lambda_1} - c)\}' = \{(p + \frac{\lambda}{\lambda_1} - c)\}' = \{(p + \frac{\lambda}{\lambda_1} - c)\}' = \{(p + \frac{\lambda}{\lambda_1} - c)\}^{'2} \{\frac{\partial^2 g}{\partial p^2} \frac{\partial g}{\partial p} + \frac{\partial^2 g}{\partial p^2} \frac{\partial g}{\partial p} \} \}. Hence, \( \frac{\partial p(\rho_0)}{\partial \rho_0} = \frac{\partial p(\rho_0)}{\partial \rho_0} = \frac{\partial^2 g}{\partial p^2} \frac{\partial g}{\partial p} \}, \) where \( \Delta \) is as defined before. Note that we have already shown that \( \Delta > \lambda (\sigma^2 - c_1 \lambda^2)(3c_1 \lambda^2 - 2\sigma^2) \{(p + \frac{\lambda}{\lambda_1} - c)\} > 0 \). We can then easily prove that \( 0 < p'(\rho_0) < \frac{\lambda\lambda'}{2 \lambda_1^2 c_1 \lambda^2} \). In the following we will show the
unimodality of \( \pi_0(\rho_0, p(\rho_0)) \). If \( p(\rho_0) \in (p^C, P^u) \), from the definition of \( p(\rho_0) \) we have
\[
\frac{d \pi_0(\rho_0, p(\rho_0))}{d \rho_0} = -K c_1 \frac{\partial g}{\partial \rho_0} + \sigma_L [b_0 - (h_0 + b_0) \phi],
\]
and
\[
\frac{d^2 \pi_0(\rho_0, p(\rho_0))}{d \rho_0^2} = -K c_1 \frac{\partial^2 g}{\partial p \partial \rho_0} + \frac{\partial^2 g}{\partial p \partial \rho_0} \rho'_0(\rho_0) - \sigma_L (h_0 + b_0) \phi.
\]
For proving unimodality of \( \pi_0(\rho_0, p(\rho_0)) \) we only need to show that \( \frac{d^2 \pi_0(\rho_0, p(\rho_0))}{d \rho_0^2} \) | \( \frac{d \pi_0(\rho_0, p(\rho_0))}{d \rho_0} = 0 \) < 0.

From \( \frac{d \pi_0(\rho_0, p(\rho_0))}{d \rho_0} = 0 \) it is clear that \( K c_1 \frac{\partial g}{\partial \rho_0} = \sigma_L [b_0 - (h_0 + b_0) \phi] \). Substituting this in \( \frac{d^2 \pi_0(\rho_0, p(\rho_0))}{d \rho_0^2} \) we get
\[
\frac{d^2 \pi_0(\rho_0, p(\rho_0))}{d \rho_0^2} \bigg| \frac{d \pi_0(\rho_0, p(\rho_0))}{d \rho_0} = 0 \bigg) = \frac{-\sigma_L [b_0 - (h_0 + b_0) \phi]}{\frac{\partial g}{\partial \rho_0}} \left( \frac{\partial^2 g}{\partial p \partial \rho_0} + \frac{\partial^2 g}{\partial p \partial \rho_0} \rho'_0(\rho_0) - \sigma_L (h_0 + b_0) \phi \right). \]
This implies that proving \( \frac{d^2 \pi_0(\rho_0, p(\rho_0))}{d \rho_0^2} < 0 \) is equivalent to proving
\[
[b_0 - (h_0 + b_0) \phi] \left( \frac{\partial^2 g}{\partial p \partial \rho_0} + \frac{\partial^2 g}{\partial p \partial \rho_0} \rho'_0(\rho_0) \right) + (h_0 + b_0) \phi \frac{\partial g}{\partial \rho_0} > 0, \tag{9}
\]
since \( \frac{\partial g}{\partial \rho_0}, \sigma_L > 0 \). As \( b_0 - (h_0 + b_0) \phi > 0, \frac{\partial^2 g}{\partial p \partial \rho_0} < 0 \) and \( 0 < \rho'_0(\rho_0) < \frac{\lambda v'}{2\lambda^2 (\sigma^2 - c_1 \lambda^2)}, \) in order to prove that the inequality in (9) is true, it is sufficient to demonstrate that
\[
[b_0 - (h_0 + b_0) \phi] \left( \frac{\partial^2 g}{\partial p \partial \rho_0} + \frac{\partial^2 g}{\partial p \partial \rho_0} \rho'_0(\rho_0) \right) + (h_0 + b_0) \phi \frac{\partial g}{\partial \rho_0} > 0. \tag{10}
\]
Substituting the expressions for \( \frac{\partial^2 g}{\partial p \partial \rho_0}, \frac{\partial^2 g}{\partial p \partial \rho_0} \) and \( \frac{\partial g}{\partial \rho_0} \) in the above expression and simplifying, it is rather straightforward to show that (10) is equivalent to proving
\[
[b_0 - (h_0 + b_0) \phi] \left\{ \frac{\lambda^2 v''}{2\sigma^3} - \frac{\lambda^2 v'^2}{4\sigma^3 (\sigma^2 - c_1 \lambda^2)} \right\} + (h_0 + b_0) \phi \frac{\lambda^2 v'}{2\sigma^3} < 0. \tag{11}
\]
Since \( b_0 - (h_0 + b_0) \phi > 0 \) and \( \frac{\lambda^2 v'^2}{4\sigma^3 (\sigma^2 - c_1 \lambda^2)} > 0 \), if we just show that
\[
[b_0 - (h_0 + b_0) \phi] v'' + (h_0 + b_0) \phi v' = b_0 [(1 - \Phi) v'' + \Phi v'] + h_0[\phi v' - \Phi v''] < 0, \tag{12}
\]
then by backtracking we can see that (9) is true, and hence \( \pi_0(\rho_0, p(\rho_0)) \) is unimodal.

So if \( b_0 [(1 - \Phi) v'' + \phi v'] + h_0[\phi v' - \Phi v''] < 0 \), then \( \pi_0(\rho_0, p(\rho_0)) \) is unimodal. Note from Lemma 4(c) that
\[(1 - \Phi) v'' + \phi v' < 0 \] below some \( \alpha > 0 \) and non-negative after that, and Lemma 4(b) indicates that \( \phi v' - \Phi v'' < 0 \). There are two cases to consider in order to prove the validity of inequality (12).

**Case 1** \((1 - \Phi) v'' + \phi v' \leq 0\): In this case, clearly \( b_0 [(1 - \Phi) v'' + \phi v'] + h_0[\phi v' - \Phi v''] < 0 \).

**Case 2** \((1 - \Phi) v'' + \phi v' > 0\): In this case in order to prove \( b_0 [(1 - \Phi) v'' + \phi v'] + h_0[\phi v' - \Phi v''] < 0 \), we need to prove \( h_0 < \frac{\Phi v'' - \phi v'}{(1 - \Phi) v'' + \phi v'} \), (note that for the right-hand side the numerator is positive based on Lemma 4(b) and the denominator is also positive for this particular case). On the other hand, we can numerically show that
if \( [(1 - \Phi)v'' + \phi v'] > 0 \), then \( \frac{\Phi v'' - \phi v'}{(1 - \Phi)v'' + \phi v'} > 105 \). It is important to point out that \( \frac{\Phi v'' - \phi v'}{(1 - \Phi)v'' + \phi v'} \) is only a function of \( \rho_0 \) in terms of \( \phi \) and \( \Phi \). This is so because if we divide the numerator and the denominator by \( \sigma_{L0}^2 \), then they both involve \( \frac{v'(\rho_0)}{\sigma_{L0}^2} \) and \( \frac{\phi v'}{\sigma_{L0}^2} \). We have already shown (in the proof of Lemma 1) that \( \frac{v'(\rho_0)}{\sigma_{L0}^2} \) and \( \frac{\phi v'}{\sigma_{L0}^2} \) are only functions of \( \rho_0 \) in terms of \( \phi \) and \( \Phi \). This implies that \( \frac{\Phi v'' - \phi v'}{(1 - \Phi)v'' + \phi v'} \) is only a function of \( \rho_0 \). Hence, we can check fairly accurately the numerical value of \( \frac{\Phi v'' - \phi v'}{(1 - \Phi)v'' + \phi v'} \) using standard mathematical software. Now if we assume that \( \frac{b_0}{\rho_0} \leq 105 \), then clearly \( \frac{b_0}{\rho_0} < \frac{\Phi v'' - \phi v'}{(1 - \Phi)v'' + \phi v'} \). Consequently, even if \( [(1 - \Phi)v'' + \phi v'] > 0 \), \( b_0[(1 - \Phi)v'' + \phi v'] + h_0[\phi v' - \Phi v''] < 0 \).

Since \( b_0[(1 - \Phi)v'' + \phi v'] + h_0[\phi v' - \Phi v''] < 0 \) for \( \frac{b_0}{\rho_0} \leq 105 \), \( \pi_0(\rho_0, p(\rho_0)) \) is unimodal for \( p(\rho_0) \in (p^C, P^u) \) under that condition. If either \( p(\rho_0) = p^C \) or \( p(\rho_0) = P^u \), we can also show the unimodality of \( \pi_0(\rho_0, p(\rho_0)) \) over the corresponding intervals in a similar fashion. Since the first derivative of \( \pi_0(\rho_0, p(\rho_0)) \) is continuous, \( \pi_0(\rho_0, p(\rho_0)) \) is unimodal over \( \rho_0 \in (-\infty, \rho_0^D) \).

For \( Var(L) = 0 \), evidently the optimal \( \rho_0, \rho_0^D = \Phi^{-1}(\frac{b_0}{\rho_0 + b_0}) \). Hence, \( \pi_0(w(p), \rho_0) \) can be simplified as \( \pi_0(p) = [p + \frac{\lambda(p)}{\sigma_{L0}^2} - c] \lambda(p) - (h_0 + b_0)\phi(\rho_0^D)\sigma_{L0} \). It is then relatively simple (from \( \frac{\partial^2 \pi_0(p)}{\partial p^2} \)) to show that if \( \lambda(p) \) is decreasing, concave, and \( 2\lambda'\lambda'' + \lambda\lambda''' \geq 0 \) \( \forall p \in [p^C, P^u] \), then \( \pi_0(p) \) is concave \( \forall p \in [p^C, P^u] \).

**Proof of the Effects of the Characteristics on the Decision Variables for \( y(p) = A - Bp \)**

Note that \( p^C \) satisfies \( \{(p - c)\lambda - K\sigma(\rho_0^C, p)'\}|_{p = p^C} = 0 \), which simplifies to

\[
(p^C - c)\lambda' + \lambda - K \frac{c_1 \lambda}{\sigma} = 0.
\]

(13)

On the other hand, \( \rho_0^C \) satisfies \( \left\{ -K\sigma(\rho_0^C, p^C) - \sigma_{L0} h_0[\rho_0^C \Phi + \phi] \right\}|_{\rho_0 = \rho_0^C} = 0 \), which is equivalent to

\[
-K \frac{v'}{2\sigma} - \sigma_{L0} h_0 \Phi = 0.
\]

(14)

Let \( M = \begin{pmatrix}
\frac{\partial^2 \Pi}{\partial p^2} | \frac{\partial^2 \Pi}{\partial p_0^2 \partial p'} \\
\frac{\partial^2 \Pi}{\partial p_0^2 \partial p'} | \frac{\partial^2 \Pi}{\partial p_0^2} | \frac{\partial^2 \Pi}{\partial p_0^2} | \frac{\partial^2 \Pi}{\partial p_0^2}
\end{pmatrix} \) and its inverse \( M^{-1} = \frac{1}{|M|} \begin{pmatrix}
\frac{\partial^2 \Pi}{\partial (p^2)^2} & -\frac{\partial^2 \Pi}{\partial p \partial p'} \\
\frac{\partial^2 \Pi}{\partial p_0 \partial p'} & \frac{\partial^2 \Pi}{\partial p_0^2} | \frac{\partial^2 \Pi}{\partial p_0^2} | \frac{\partial^2 \Pi}{\partial p_0^2} | \frac{\partial^2 \Pi}{\partial p_0^2}
\end{pmatrix} \)

Taking derivatives with respect to \( B \) on both sides of (13) and (14), we get (after substituting the expressions for \( \frac{\partial^2 \Pi}{\partial (p^2)^2}, \frac{\partial^2 \Pi}{\partial p \partial p'} \) and \( \frac{\partial^2 \Pi}{\partial (p^2)^2} \) and simplifying)

\[
\left( \begin{array}{c}
\frac{\partial \Pi}{\partial B} \\
\frac{\partial \Pi}{\partial B_0}
\end{array} \right) = M^{-1} \begin{pmatrix}
p^C - \frac{\lambda}{\sigma} - K \frac{c_1 \lambda^2}{\sigma^2} \frac{p^C - c_1 \lambda^2}{\sigma^3} \\
K \frac{c_1 \lambda}{\sigma^2} p^C \frac{c_1 \lambda}{\sigma^3} \\
K \frac{c_1 \lambda}{\sigma^2} p^C \frac{c_1 \lambda}{\sigma^3}
\end{pmatrix}
\]

Based on the above expression, we get

\[
\frac{\partial p^C}{\partial B} |_{M} < \frac{K v'^2}{4\sigma^3} \left( \frac{(c_2 + c_1 \lambda^2) 2(v' \phi - v'' \Phi) v (p^C - \frac{\lambda}{\sigma} - K \frac{c_1 \lambda^2}{\sigma^3})}{\Phi v'^2} - \frac{K (c_1 \lambda)^2 \lambda P^C}{\sigma^3} \right)
\]

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< \frac{K v^2}{4 \sigma^3} \left\{ -c_1 \lambda^2 \frac{-K c_1 \lambda p^C \frac{\sigma^2 - \sigma^2}{\sigma^2}}{v} - \frac{K (c_1 \lambda)^2 \lambda' p^C}{\sigma^3} \right\} = \frac{K^2 v^2 (c_1 \lambda)^2 \lambda' p^C}{4 \sigma^6 v} \lambda' < 0,

\text{and} \quad \frac{\partial \rho^C}{\partial B} |M| = \frac{K c_1 \lambda \nu'}{2 \sigma^3} \left[ \lambda' p^C + \lambda \right] = \frac{K c_1 \lambda \nu'}{2 \sigma^3} \left[ c \lambda' + \frac{K c_1 \lambda \nu'}{\sigma} \right] > 0.

The last equality is obtained by using (13). Hence, \( \frac{\partial \rho^C}{\partial B} < 0 \) and \( \frac{\partial \rho^C}{\partial \nu} > 0 \). Similarly, we can show that \( \frac{\partial \rho^C}{\partial \sigma_d} < 0 \), \( \frac{\partial \rho^C}{\partial \sigma_a} > 0 \), and \( \frac{\partial \rho^C}{\partial \nu (\nu (L))} > 0 \) and \( \frac{\partial \rho^C}{\partial \sigma_d} > 0 \).

For the decentralized system, note that \( p^D \) satisfies \( \{ [p + \frac{\lambda}{\nu} - K c_1 \frac{\lambda}{\sigma (\rho_0^D, v)} - c] \lambda'] |_{\rho = p^D} = 0 \), i.e.,

\[
(p - c) \lambda' + 3 \lambda - K c_1 \frac{2 \sigma^2 - \sigma^2}{\sigma^3} \lambda \lambda' = 0.
\]

Also \( \rho_0^D \) satisfies \( \{-K c_1 \frac{\lambda^2}{\sigma (\rho_0^D, p)} + \sigma L_0 [b_0 \rho_0 - (h_0 + b_0) (\rho_0 \Phi - \Phi)] \} |_{\rho_0 = \rho_0^D} = 0 \), which simplifies to

\[
K c_1 \lambda' p^C \left( \frac{\lambda}{\nu} \right) \lambda' + \sigma L_0 [b_0 - (h_0 + b_0) \Phi (\rho_0^D)] = 0.
\]

Let \( N = \left( \begin{array}{ccc} \frac{\partial^2 \pi_0}{\partial p^D} & \frac{\partial^2 \pi_0}{\partial p^D \partial \rho^D} \\ \frac{\partial^2 \pi_0}{\partial p^D \partial \rho^D} & \frac{\partial^2 \pi_0}{\partial \rho^D} \end{array} \right) \) and its inverse \( N^{-1} = \frac{1}{|N|} \left( \begin{array}{ccc} \frac{\partial^2 \pi_0}{\partial p^D \partial \rho^D} & \frac{\partial^2 \pi_0}{\partial \rho^D \partial p^D} \\ \frac{\partial^2 \pi_0}{\partial \rho^D \partial p^D} & \frac{\partial^2 \pi_0}{\partial p^D \partial \rho^D} \end{array} \right) \)

It is easy to check that

\[
\frac{\partial^2 \pi_0}{\partial (\rho_0^D)^2} = \frac{K c_1 \lambda^2}{4 \sigma^5 [b_0 - (h_0 + b_0) \Phi]} \{ 2 \sigma^2 [b_0 - (h_0 + b_0) \Phi] v'' + 2 \sigma^2 (h_0 + b_0) \phi v' - 3 \sigma^2 [b_0 - (h_0 + b_0) \Phi] \} < 0,
\]

\[
\frac{\partial^2 \pi_0}{\partial p^D \partial \rho^D} = K c_1 \lambda \lambda' p^C \left( \frac{2 \sigma^2 - 3 \sigma^2}{2 \sigma^3} \right) > 0, \quad \text{and} \quad \frac{\partial^2 \pi_0}{\partial (p^D)^2} = 4 \lambda - K c_1 \lambda^2 \left( \frac{\sigma \lambda^2}{\sigma^3} \right) (2 \sigma^2 - 3 \sigma^2) \lambda^2 < 0.
\]

Taking derivatives on both sides of (15) and (16) with respect to \( B \), we get

\[
\left( \begin{array}{c} \frac{\partial \rho^D}{\partial B} \\ \frac{\partial \rho^D}{\partial B} \\ \frac{\partial \rho^D}{\partial B} \end{array} \right) = N^{-1} \left( \begin{array}{c} 3 \sigma^2 - 3 \lambda^2 \\ -K c_1 \lambda p^C \left( \frac{\sigma^2 - \sigma^2}{\sigma^3} \right) (2 \sigma^2 - 3 \sigma^2) \lambda^2 \end{array} \right).\]

Based on the above expression, we get

\[
\frac{\partial \rho^D}{\partial B} |N| = -K c_1 \lambda \lambda' p^C \left( \frac{3 \sigma^2 - 3 \lambda^2}{2 \sigma^5} \right) (3 \sigma^2 - 3 \lambda^2) + 4 \lambda' K c_1 \lambda \lambda' p^C \left( \frac{3 \sigma^2 - 3 \lambda^2}{2 \sigma^5} \right) (3 \sigma^2 - 3 \lambda^2) > 0,
\]

\[
\frac{\partial \rho^D}{\partial B} |N| = K c_1 \lambda \lambda' p^C \left( \frac{3 \sigma^2 - 3 \lambda^2}{2 \sigma^5} \right) (3 \lambda \lambda' + p^D) > 0.
\]

Hence, \( \frac{\partial \rho^D}{\partial B} < 0 \), \( \frac{\partial \rho^D}{\partial B} > 0 \). Similarly, we can show that \( \frac{\partial \rho^D}{\partial \nu (\nu (L))} > 0 \), \( \frac{\partial \rho^D}{\partial \sigma_d} > 0 \), and \( \frac{\partial \rho^D}{\partial \sigma_d} > 0 \).
Proof of Lemma 3

We know that $\rho^D = \rho^C = \Phi^{-1}(\frac{b}{b+h})$ for any $Var(L)$. By substituting $\rho^D$ and $\rho^C$ into the retailer’s and the centralized system profit function, respectively, we can express those functions in terms of $p$ and $\rho_0$. Note that if $Var(L) = 0$, then the riskless part and the uncertainty-related inventory cost part of the centralized system profit as well as those of the channel members in the decentralized chain are separable (refer to Section 4). The riskless part is independent of $\rho_0$, while the cost portion is independent of $p$. Based on the RWP contract, the distributor will receive a $\beta$ share of the retailer’s revenue, and $w = (1 - \beta)c$. Hence, the riskless part of the distributor’s profit $= (w - c)\lambda(p) + \beta p \lambda(p) = \beta(p - c)\lambda(p)$, and the retailer’s riskless profit $= (1 - \beta)(p - c)\lambda(p)$. Moreover, the centralized system’s riskless profit $= (p - c)\lambda(p)$. Obviously, for a given $\beta$, then $p^D = p^C$, and the sum of the riskless profits of the channel members is equal to that of the centralized system. As far as the uncertainty-induced inventory cost part is concerned, if $b_0 = b_0^* = h_0 \frac{\Phi(\rho^C)}{1-\Phi(\rho^C)}$, then $\rho^D_0 = \rho^C_0$ (Shang and Song 2003), and the sum of the cost part of the channel members is equal to that of the centralized system. Hence, with the RWP contract, $\pi^D_0 + \pi^D = \Pi^C$. ♦

References


