Quality-Based Competition, Profitability, and Variable Costs

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Appendix

Equivalence of Utility Specifications. Consider (i) the utility function \( U(q, p : \theta) = q - \theta p \) where \( \theta \) has the cumulative distribution function \( F_\theta(\bullet) \) corresponding to a uniform distribution on \([\varepsilon, 1]\) versus (ii) the “classical” utility function \( U(q, p, \alpha) = \alpha q - p \) where \( \alpha \) has the (unknown) cumulative distribution \( F_\alpha(t) \). The infinitesimal \( \varepsilon \) is introduced to avoid division by 0, although taking the limit \( \varepsilon \to 0 \) poses no difficulties in our subsequent analysis. The two utility specifications are equivalent if and only if the proportion of the population preferring one product over another in our model equals the proportion of the population having identical preferences in the classical model. Straightforward algebra shows that these proportions are indeed equal if and only if \( \Pr_\alpha(\alpha \geq t) = \Pr_\theta(\theta \leq 1/t) \). However,

\[
1 - F_\alpha(t) = \Pr(\alpha : \alpha \geq t) = \Pr(\theta : \theta \leq 1/t) = F_\theta(1/t) = \begin{cases} 
1 & t < 1 \\
1/t - \varepsilon & 1 \leq t \leq 1/\varepsilon \\
1 - \varepsilon & t > 1/\varepsilon \\
0 & t > 1/\varepsilon 
\end{cases}
\]

Differentiating each side with respect to \( t \) implies the equivalent density for the taste parameter in the classical model is \( f_\alpha(t) = \frac{1}{1 - \varepsilon^2} \) for \( 1 \leq t \leq 1/\varepsilon \) and 0 elsewhere. In the special case where \( \varepsilon \to 0 \), the equivalent density for the taste parameter in the classical model is \( f_\alpha(t) = 1/t^2 \) on \([1, \infty)\).

Proposition 1. Condition 1. The terms \( q_H, q_L, \) and \( p_L \) are considered fixed (given). Suppose \( p_H < (p_L / q_L) q_H \). Then we must have \( q_L / p_L < q_H / p_H < (q_H - q_L) / (p_H - p_L) \) (see Figure 1a) and H’s profit is \( \pi_H = \min(q_H / p_H, 1)(p_H - c_H) \). It is easy to show that H’s profit is strictly increasing on this interval. Thus \( p_H \geq (p_L / q_L) q_H \), which is Condition 1. Condition 2. By Condition 1, we must have \( (q_H - q_L) / (p_H - p_L) < q_H / p_H \leq q_L / p_L \). For \( p_H \) such that \( (q_H - q_L) / (p_H - p_L) > 1 \) (see Figure 1c), H’s profit is \( \pi_H = 1 \cdot (p_H - c_H) \), which is strictly increasing. Therefore, H will always respond with a price such that \( (q_H - q_L) / (p_H - p_L) \leq 1 \) or equivalently \( p_H \geq p_L + q_H - q_L \).

Best Response for H. Because \( p_H \) must satisfy proposition 1, we have

\[
\pi_H(p_H) = \left( \frac{q_H - q_L}{p_H - p_L} \right)(p_H - c_H) \quad \text{and} \quad \tilde{\partial} \pi_H / \tilde{\partial} p_H = \frac{q_H - q_L}{p_H - p_L} \left( \frac{c_H - p_L}{p_H - p_L} \right).
\]
If \( p_L < c_h \), the function \( \pi_h \) is strictly increasing and thus “\( p_h = \infty \)” (this price is simply large and finite if one takes \( \theta \in [\varepsilon, 1] \) for \( \varepsilon > 0 \) and small).

If \( p_L = c_h \), then H’s profit is \( q_h - q_L \), regardless of H’s price provided it is feasible (see Figure 2). Thus H selects any price satisfying \( p_h \geq \max(c_h + q_h - q_L, c_h q_h / q_L) \).

If \( p_L > c_h \), then \( \pi_h \) is strictly decreasing and H will set the lowest possible feasible price, which is \( \max(p_L + q_h - q_L, p_L q_h / q_L) \). In summary, H’s best response is

\[
R_h(p_L) = \begin{cases} 
    p_H = \infty & p_L < c_h \\
    p_H = \max(c_h + q_h - q_L, c_h q_h / q_L) & p_L = c_h \\
    p_H = \max(p_L + q_h - q_L, p_L q_h / q_L) & p_L > c_h
\end{cases}
\]

**Proposition 2.** Given \( 0 < q_L < q_h \) let L chose a price \( c_L \), and H chose his best response. Thus, the total size of the served market is 1 and \( \pi_L = \left( 1 - \frac{q_h - q_L}{p_h - p_L} \right) (p_L - c_L) \). Since \( c_h > c_L \) has positive market share and profits as long as \( 1 > \frac{q_h - q_L}{p_h - c_h} \). This is easily verified by direct substitution.

The following lemma is needed to simplify the analysis of L’s best response. It is based on straightforward algebra and therefore offered without proof.

**Lemma 1.** If \( p_L \) satisfies \( p_L \leq c_L \) (a “market covering” price), then Condition 2 of proposition 1 implies Condition 1. If \( p_L > c_L \), then Condition 1 implies Condition 2.

**Best response for L.** Market is covered case. On the interval \( p_L \leq q_L \), L covers the market and the resulting profit function is \( \pi_L = \left( 1 - \frac{q_h - q_L}{p_h - p_L} \right) (p_L - c_L) \). Assuming the market is covered implicitly requires \( c_L \leq q_L \); if this is not the case, the market cannot be covered and one proceeds directly to the analysis of the uncovered market. For the covered market case, we have

\[
\frac{\partial \pi_L}{\partial p_L} = \left[ 1 - \frac{q_h - q_L}{(p_h - p_L)^2} \right] (p_h - c_L).
\]

The equation \( \frac{\partial \pi_L}{\partial p_L} = 0 \) has two roots. However, only one of these roots is less than \( p_h \), and therefore it is the only relevant response. The root is

\[
p_L^* = p_h - \sqrt{(p_h - c_L)(q_h - q_L)}.
\]

Because of H’s price floor—\( p_H \geq \max(c_h, c_L + q_h - q_L, c_L q_h / q_L) \)—one can show by direct calculation that (i) \( p_L^* \) is an increasing function of \( p_H \), (ii) \( p_L^* \geq c_L \), and (iii) Condition 2 of
propostion 1 holds (observe that we must still show L’s price response forms a feasible pair so that the initial profit representation for L is valid.)

If the root additionally satisfies \( p_L^u \leq q_L \), then \( (p_H, p_L^u) \) must be a feasible pair of prices by Lemma 1. In this case, we may make the stronger statement that \( p_L^u \) is a global maximum for all \( p_L < p_H \). This follows from the fact that the function \( \left( 1 - \frac{q_H - q_L}{p_H - p_L} \right)(p_L - c_L) \) is concave on the extended interval \( p_L < p_H \) and vanishes at most once on this interval. Thus for feasible prices satisfying \( p_L > q_L \), we observe

\[
\left( 1 - \frac{q_H - q_L}{p_H - p_L} \right)(p_L^u - c_L) \geq \left( 1 - \frac{q_H - q_L}{p_H - p_L} \right)(p_L - c_L) \geq \left( \frac{q_L - q_H - q_L}{p_L - p_L} \right)(p_L - c_L)
\]

The latter inequality ensures that \( p_L^u \) is a global maximum for \( \pi_L \) if \( p_L^u \leq q_L \).

If \( p_L^u > q_L \), then the derivative of \( \pi_L \) is positive on the interval \( p_L \leq q_L \) and so \( \pi_L \) is increasing on this interval. The maximum (and thus L’s best response) occurs at \( p_L = q_L \). This response forms a feasible pair because (a) if Condition 2 of proposition 1 is satisfied for \( p_L^u \) then it is satisfied for \( p_L = q_L \) and (b) Condition 2 and Condition 1 are identical if \( p_L = q_L \).

**Market not covered case.** If L has a positive market share and \( p_L > q_L \), the market is not covered. Observe that \( p_L > q_L \) can only occur if \( p_H > q_H \) (otherwise the market is already covered by H — see Figure 1b and Figure 2). Additionally, we may assume H has priced above his price floor, \( p_H \geq \text{max}(c_H, c_L + q_H - q_L, c_L \frac{q_H}{q_L}) \). For \( p_L \) satisfying \( p_L \geq q_L \), L’s profit function is given by \( \pi_L = \left( \frac{q_L - q_H - q_L}{p_L - p_L} \right)(p_L - c_L) \), and \( \frac{\partial \pi_L}{\partial p_L} = \left[ \frac{q_H - q_L}{p_H - p_L} \right](c_L - p_H) + \frac{c_L q_H}{p_L} \). If the derivative vanishes at an interior point \( p_L > q_L \), then it is a global maximum on the interval \( p_L \geq q_L \) since \( \pi_L \) is strictly concave on this interval. Again, there are two possible roots for \( \frac{\partial \pi_L}{\partial p_L} = 0 \). However, in this case only one root is positive:

\[
p_L^* = \frac{c_L q_H}{(q_H - q_L)(p_H - c_L)} \cdot p_H
\]

It can be shown (we omit the tedious algebra) that \( (p_H, p_L^*) \) satisfies Condition 1 of proposition 1 provided \( p_H \geq c_L \frac{q_H}{q_L} \). Note that the latter condition is guaranteed by H’s price floor.
Moreover, it can be shown the root is an increasing function of $p_H$ on the interval $p_H \geq \max(c_H, c_L + q_H - q_L, c_L \frac{q_H}{q_L})$ and further satisfies $p_L^{**} \geq c_L$.

If $p_L^{**} \geq q_L$, then Lemma 1 implies Condition 2 must hold as well. Thus $(p_H, p_L^{**})$ is a feasible pair of prices whenever $p_L^{**} \geq q_L$. This solution is also a global maximum for $\pi_L$ for reasons explained next. We observe first that the derivative for $\pi_L$ cannot vanish on both of the intervals $p_L \leq q_L$ and $p_L > q_L$. For suppose $\frac{\partial \pi_L}{\partial p_L} = 0$ for $x^* \leq q_L$, then we have

$$\left(1 - \frac{q_H - q_L}{p_H - x^*}\right)(x^* - c_L) \geq \left(1 - \frac{q_H - q_L}{p_H - p_L}\right)(p_L - c_L) \geq \left(\frac{q_L - q_H}{p_L - p_L}\right)(p_L - c_L)$$

for all feasible prices $p_L \geq q_L$. Observe that the two profit expressions agree at the crossover point $p_L = q_L$. However, the actual concave profit function that applies on the interval $p_L \geq q_L$, $\pi_L = \left(\frac{q_L - q_H}{p_L - p_L}\right)(p_L - c_L)$, is bounded above by the strictly decreasing concave function $\left(1 - \frac{q_H - q_L}{p_H - p_L}\right)(p_L - c_L)$. This prohibits $\frac{\partial \pi_L}{\partial p_L}$ from vanishing on the interval $p_L > q_L$ (if it has already vanished for $x^* \leq q_L$). Thus if $p_L^{**} > q_L$, $\pi_L$ must be increasing on $[c_L, q_L]$, increasing on $[q_L, p_L^{**}]$, and then decreasing thereafter. This makes $p_L^{**}$ a global maximum.

If the solution satisfies $p_L^{**} \leq q_L$, then the profit function is decreasing on the interval $p_L \geq q_L$ and the maximum on this interval occurs at the endpoint $p_L = q_L$, which is feasible since $p_H > q_H$ (see Figure 2). It can be shown that $p_L^{**} < p_L^*$ provided H chooses $p_H$ above his price floor. Consequently, the best response for L is

$$R_L(p_H) = \begin{cases} p_L = p_L^* & \text{if } p_L^* < q_L \\ p_L = p_L^{**} & \text{if } p_L^{**} > q_L \\ p_L = q_L & \text{otherwise} \end{cases}$$

**Theorem 1: Price Equilibrium.** *Existence of a price equilibrium.* We observe that the best response curves always intersect. This can be shown in three steps, whose details are left to the reader. Step 1: The minimum point on H’s response curve has coordinates $p_L = c_H$, $p_H = \max(c_H + q_H - q_L, c_H \frac{q_H}{q_L})$. This point occurs at the bottom of a vertical line segment positioned at $p_L = c_H$ (see Figure 3). Step 2. If the minimum response on H’s curve is $p_H = c_H + q_H - q_L$, then $p_L^*(c_H + q_H - q_L) < c_H$, which implies $R_L(c_H + q_H - q_L) < c_H$. Step 3. If
the minimum response on H’s curve is \( p_H = c_H \frac{q_H}{q_L} \), then this implies \( q_L < c_H \). One can then show \( p_L^*(c_H \frac{q_H}{q_L}) < q_L < c_H \), which implies \( R_L(c_H \frac{q_H}{q_L}) \leq q_L < c_H \). Since \( R_L(p_H) \) is an increasing function satisfying \( R_L(p_H) \uparrow \infty \) as \( p_H \uparrow \infty \), it must cross the infinite vertical segment of H’s response curve (see Figure 3) at some point above \( p_H = \max(c_H + q_H - q_L, c_H \frac{q_H}{q_L}) \). Observe that a simple perturbation argument implies the existence of a similar intersection for the case where \( \theta \in [\varepsilon,1] \), provided \( \varepsilon \) is sufficiently small.

The price equilibrium. The only potential equilibrium solution occurs when \( p_L = c_H \). In this case, the profit function becomes \( \pi_H = \left( \frac{q_H - q_L}{p_H - c_H} \right)(p_H - c_H) = q_H - q_L \). Consequently, \( H \) appears indifferent to any feasible price \( p_H \) since they all result in an identical profit of \( q_H - q_L \). (this indifference disappears for \( \theta \in [\varepsilon,1] \) with \( \varepsilon > 0 \) and small). However, \( H \) must still choose his price carefully so that \( L \) accepts the price \( p_L = c_H \) and has no incentive to change. This is indeed the case if \( p_H \) is set so that \( p_L = c_H \) is \( L \)'s optimal response. Analysis of the first term in \( L \)'s response curve shows that if \( c_H < q_L \) then the value of \( p_H \) which drives \( p_L \) to \( c_H \) is
\[
p_H^* = c_H + 1/2(q_H - q_L) + \sqrt{1/4(q_H - q_L)^2 + (c_H - c_L)(q_H - q_L)}
\]
One can readily check that \( p_H^* \) as defined above and \( p_L^* = c_H \) form a feasible pair of prices (this ensures that our original profit representations are valid). These are the equilibrium prices when \( c_H < q_L \). Analysis of the second term in \( L \)'s response curve shows that if \( c_H > q_L \), the appropriate selection is
\[
p_H^* = c_H + \alpha/2 + \sqrt{\alpha^2/4 + \alpha(c_H - c_L)} \quad \text{where} \quad \alpha = \frac{(q_H - q_L)c_H^2}{c_L q_L}
\]
One can check that \( p_H^* \) as defined above and \( p_L^* = c_H \) form a feasible pair of prices. These are the equilibrium prices when \( c_H > q_L \). Finally, if \( c_H = q_L \), then there are an infinity of price equilibrium solutions. One can calculate a value for \( p_H^* \) using either of the preceding formulas, although any price selected between these two values will also suffice. This occurs because the finite vertical segment in \( L \)'s best response curve perfectly coincides with the infinite vertical segment in \( H \)'s response curve (see Figure 3). We note that this situation does not occur when the price sensitivity parameter satisfies \( \theta \in [\varepsilon,1] \) \( \varepsilon > 0 \). The solution with the lowest price for \( H \) is the limiting price as \( \varepsilon \rightarrow 0 \).

**Theorem 2:** The result follows from direct substitution into equation (2) using the results from Theorem 1.

**Theorem 3:** Since \( q_L < q_H \) by definition, the market is necessarily uncovered unless \( p_L^* = c_H \leq q_H \), which is therefore the only situation to assume. For notational convenience,
recall $c_H = c(q_H)$. The proof proceeds by showing that L’s market share for the uncovered market as $q_L \uparrow c_H$ (see (10)) exceeds L’s market share for the entire covered market $q_L \geq c_H$ (see (9)). Since profit margins are also larger on the uncovered market side, it follows that the profit for L as $q_L \uparrow c_H$ (see 10) will dominate all profits for (9) with $q_L \geq c_H$.

According to (10), the market share for L as $q_L \uparrow c_H$ is

$$\lim_{q_L \uparrow c_H} \left( \frac{q_L}{c(q_H)} \right) = \frac{2c(q_L)}{1 + \frac{4c(q_L)c_H[c(q_H) - c(q_L)]}{c(q_H)q_H - c_H}}.$$ 

Since $\left( \frac{c(q_H)}{c(c_H)} \right) > 1$,

$$\lim_{q_L \uparrow c_H} \left( \frac{q_L}{c(q_H)} \right) = \frac{2}{1 + \frac{4c(q_H)c_H[c(q_H) - c(c_H)]}{c(c_H)q_H - c_H}}.$$ 

Because $c(q)/q$ is log-concave, so is $c(q)$. Thus for any $y \geq x$ we must have $\frac{c'(y)}{c(y)} \geq \frac{c'(x)}{c(x)}$.

Because $c(q)$ is convex, for any $y \geq x$ we must have $\frac{c(y) - c(x)}{y - x} \geq c'(x)$, where the difference quotient at $y = x$ is interpreted as $c'(x)$. Thus for $y = q_H, x = c_H \leq q_H$ and $z \leq q_H$ we have

$$\frac{c(q_H)}{c(c_H)} \cdot \frac{c(q_H) - c(c_H)}{q_H - c_H} \geq c'(q_H) \geq \frac{c(q_H) - c(z)}{q_H - z},$$

where the last inequality follows from the convexity of $c(q)$. It follows that for any $z \leq q_H$,

$$\lim_{q_L \uparrow c_H} \left( \frac{q_L}{c(q_H)} \right) \geq 1 - \frac{2}{1 + \frac{4c(q_H)c_H[c(q_H) - c(c_H)]}{c(c_H)q_H - c_H}}.$$ 

The last term is precisely L’s market share in (9) for $z$ in the interval $q_H \geq z \geq c_H$. This demonstrates that the market share for L as $q_L \uparrow c_H$ exceeds that for all $q_L \geq c_H$ where (9)
applies. This implies the maximum profit occurs over the region $q_L \leq c_H$, i.e., the uncovered market region whose profit is determined by (10).

We introduce three lemmas that will help with the proof of Theorem 4.

**Lemma 2.** Suppose $c(q)/q$ is convex and log-concave. Then the ratio function $r(q) = \frac{c(q)/q}{c(q_{\text{max}}) - c(q)}$ is non-decreasing.

**Proof.** Observe that $r(q)$ is continuous on $[0, q_{\text{max}}]$ with $r(0) = q_{\text{max}} c'(0)/c(q_{\text{max}})$ and $r(q_{\text{max}}) = c(q_{\text{max}})/[c'(q_{\text{max}}) q_{\text{max}}]$. Moreover, $r(q)$ is differentiable on $(0, q_{\text{max}})$ and therefore it is non-decreasing on $[0, q_{\text{max}}]$ if and only if its derivative is nonnegative on $(0, q_{\text{max}})$. The latter is equivalent, after suitable algebraic manipulations, to the inequality condition

$$\frac{c(q_{\text{max}}) c'(q)}{q_{\text{max}} c(q)} q \geq \frac{c(q_{\text{max}}) - c(q)}{q_{\text{max}} - q}.$$

We will now show the latter inequality is true. Observe that we may write $c(q) = q h(q)$ with $h(q)$ convex and log-concave. Log-concavity of $h(q)$ implies $h'(q)/h(q)$ is a decreasing function of $q$. Therefore, beginning with the left hand side of the previous inequality condition

$$\frac{c(q_{\text{max}}) c'(q)}{q_{\text{max}} c(q)} q = h(q_{\text{max}}) \left[ 1 + q \frac{h'(q)}{h(q)} \right] \geq h(q_{\text{max}}) \left[ 1 + q \frac{h'(q_{\text{max}})}{h(q_{\text{max}})} \right] = \{ h(q_{\text{max}}) + q h'(q_{\text{max}}) \}.$$

But $h(q)$ is convex, so the final term in curly brackets satisfies

$$\{ h(q_{\text{max}}) + q h'(q_{\text{max}}) \} \geq h(q_{\text{max}}) + q \frac{h(q_{\text{max}}) - h(q)}{q_{\text{max}} - q} = \frac{q_{\text{max}} h(q_{\text{max}}) - q h(q)}{q_{\text{max}} - q} = \frac{c(q_{\text{max}}) - c(q)}{q_{\text{max}} - q}.$$

**Lemma 3.** Suppose $f(x)$ is nonnegative, differentiable, and strictly concave on $[a, b]$. Suppose $g(x)$ is differentiable, non-increasing, and positive on $[a, b]$. Let the maximum of $f(x)$ on $[a, b]$ occur at the point $x_f^*$, and let a global maximum of $f(x)g(x)$ on $[a, b]$ occur at the point $x_{fg}^*$. Then $x_{fg}^* \leq x_f^*$.

**Proof.** For $x \in (x_f^*, b]$, $f'(x) < 0$. After applying the product rule and the various sign conditions stated in the theorem, it follows that $(f g)'(x) < 0$ on this interval as well. This proves the result.

**Lemma 4.** Suppose $n(x)$ and $d(x)$ are nonnegative, continuous functions on $[a, b]$ that are
also differentiable on \((a,b)\). Assume \(n(x)\) is non-decreasing. If the ratio \(\frac{n(x)}{\alpha + d(x)}\) is continuous on \([a,b]\), non-decreasing on \([a,b]\), and differentiable on \((a,b)\), then \(\frac{n(x)}{\alpha + d(x)}\) is continuous and non-decreasing on \([a,b]\) for any \(\alpha > 0\).

**Proof.** Continuity of \(\frac{n(x)}{\alpha + d(x)}\) on \([a,b]\) is clear. Since \(\frac{n(x)}{d(x)}\) is non-decreasing and differentiable on \((a,b)\), \(n'(x)d(x) - n(x)d'(x) \geq 0\) on \((a,b)\), and so \(n'(x)[\alpha + d(x)] - n(x)d'(x) \geq 0\) on \((a,b)\), which implies \(\frac{n(x)}{\alpha + d(x)}\) is non-decreasing on \([a,b]\).

**Theorem 4.** Part (a). By lemma 2, \(r(q_L)\) is non-decreasing on \([0,q_{\text{max}}]\). Thus \(r(q)c(q)/4c(q)\) is non-decreasing (with a removable singularity at 0). The following chain of non-decreasing (non-d for short) functions is implied:

\[
\frac{r(q)c(q)}{4c(q)} = \frac{(c(q_L))^2}{4c(q_L)q_L \frac{c_{\text{max}} - c(q_L)}{q_{\text{max}} - q_L}} \quad \text{non-d} \quad \Rightarrow \quad \frac{(c(q_L))^2}{c_{\text{max}}^2 + 4c(q_L)q_L \frac{c_{\text{max}} - c(q_L)}{q_{\text{max}} - q_L}} \quad \text{non-d}
\]

\[
\Rightarrow \quad \sqrt{\frac{2c(q_L)/c_{\text{max}}}{1 + \sqrt{\frac{4c(q_L)q_L \frac{c_{\text{max}} - c(q_L)}{q_{\text{max}} - q_L}}}} \quad \text{non-d.}
\]

Consequently,

\[
1 - \frac{2c(q_L)/c_{\text{max}}}{1 + \sqrt{\frac{4c(q_L)q_L \frac{c_{\text{max}} - c(q_L)}{q_{\text{max}} - q_L}}}} = g(q_L)
\]

is seen to be positive and non-increasing. If we define \(f(q_L) = (c_{\text{max}} - c(q_L))(q_L / c_{\text{max}})\), then part (a) of Theorem 4 follows immediately from Lemma 3.

For part (b) of the theorem, observe that an upper bound on \(L\)’s profit in (12) is \(\pi_{\text{L}}^* \leq c_{H}\). If \(H\) leapfrogs \(L\), the same inequality applies where \(q_L\) represents the high quality position. Since \(q_L^* < q_K^*\), \(c(q_L^*) < c(q_K^*)\), \(H\)’s best profit from leapfrogging \(L\) is bounded
above by \( c(q_k^*) \). Part (b) follows by comparing the lower bound on H’s profit ensured by part (a) and the upper bound on profit \( c(q_k^*) \) for leapfrogging L.

For part (c), observe that the profit for L is bounded above by the expression \( q_{\text{max}} K_{[0,q_{\text{max}}]}(qc(q)) \). If H leapfrogs L, then the same bound applies for the optimal position that H takes below L: \( q_L^* K_{[0,q_L^*]}(qc(q)) \), where \( q_L^* \) is L’s current position. It is immediately clear that \( q_L^* K_{[0,q_L^*]}(qc(q)) \leq q_k^* K_{[0,q_k^*]}(qc(q)) \). We will be done if we can show that \( K_{[0,q_k^*]}(qc(q)) \leq K_{[0,q_{\text{max}}]}(qc(q)) \). The condition stated in part (c) plays an essential role.

Consider an arbitrary cost function \( c(q) \in X \). Construct the associated function \( v(\lambda, q) = \lambda - \frac{c(\lambda q)}{c(q)} \) defined for \( 0 \leq \lambda \leq 1 \) and \( 0 < q \). Then

\[
\frac{\partial v(\lambda, q)}{\partial q} = -\lambda c'(\lambda q)c(q) + c(\lambda q)c'(q) = \frac{c(\lambda q)c(q)}{[c(q)]^2} \left[ \frac{c'(q)}{c(q)} q - \frac{c'(\lambda q)}{c(\lambda q)} \lambda q \right] \geq 0.
\]

The last inequality follows because \( qc'(q)/c(q) \) is assumed to be non-decreasing, and so the bracketed term must be nonnegative, too. Because \( \frac{\partial v(\lambda, q)}{\partial q} \geq 0 \), it follows that

\[
K_{[0,q_{\text{max}}]}(c(q)) = \max_{\lambda \in [0,1]} \left( \lambda - \frac{c(\lambda q_{\text{max}})}{c(q_{\text{max}})} \right) \geq \max_{\lambda \in [0,1]} \left( \lambda - \frac{c(\lambda q_k^*)}{c(q_k^*)} \right) = K_{[0,q_k^*]}(c(q)).
\]

Since \( q \in X \) and \( c(q) \in X \), we must have \( qc(q) \in X \), and consequently the same result applies for \( qc(q) \). The maximum profit H can obtain by leapfrogging L is therefore bounded above by \( q_k^* K_{[0,q_{\text{max}}]}(qc(q)) \). Part (c) now follows by insisting that the upper bound on H’s profit for leapfrogging is no better than the upper bound on H’s profit (as previously established in part (a)) for staying at \( q_{\text{max}} \).