Soybean Inventory and Forward Curve Dynamics

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Appendix 1

Proof of the two state-variables futures pricing formula. Under Assumptions 2 and 3 describing the dynamics of the log spot price and its level of mean-reversion, the price of the futures contract $F$ satisfies the partial differential equation

$$
\frac{\partial F}{\partial t} + \kappa(v_{1t} - X_t) \frac{\partial F}{\partial X} + a_1(b_1 - v_{1t}) \frac{\partial F}{\partial v_1} + a_2(b_2 - v_{2t}) \frac{\partial F}{\partial v_2} + \frac{1}{2} \sigma_1^2 e^{2v_{1t}} \frac{\partial^2 F}{\partial X^2} + \rho_{12} \sigma_1^2 e^{2v_{1t}} \frac{\partial^2 F}{\partial X \partial v_1} + \frac{1}{2} \sigma_2^2 e^{2v_{2t}} \frac{\partial^2 F}{\partial v_1^2} + \frac{1}{2} \sigma_2^2 e^{2v_{2t}} \frac{\partial^2 F}{\partial v_2^2} = 0
$$

subject to the terminal condition $F(X_T, v_{1T}, T) = S(T) = e^{h_T + X_T}$.

The solution is known (see for instance Duffie and Kan (1996)) to be of the form

$$
F_t^T = e^{h_T + A(t,T) + B(t,T)X_t + C(t,T)v_{1t}}
$$

which, substituted in (A1), leads to the system of ordinary differential equations

$$
B' - \kappa B = 0
$$

(A2)

$$
C' + \kappa B - a_1 C = 0
$$

(A3)

$$
A' + a_1 b_1 C + \frac{1}{2} e^{2v_{1t}} \left( \sigma_2 B^2 + \sigma_1^2 C^2 + 2 \rho_{12} \sigma_1 \sigma_2 B C \right) = 0
$$

(A4)

with initial conditions $A(T,T) = 0$, $B(T,T) = 1$, $C(T,T) = 0$.

Equation (A2) and (A3) are elementary and plugging their solutions into (A4) leads to the expression of $A(t,T)$, hence of $F_t^T$.

Proof of the three state-variables futures pricing formula. In this new setting, the contingent claim $F$ satisfies the partial differential equation

$$
\frac{\partial F}{\partial t} + \kappa(v_{1t} - X_t) \frac{\partial F}{\partial X} + a_1(b_1 - v_{1t}) \frac{\partial F}{\partial v_1} + a_2(b_2 - v_{2t}) \frac{\partial F}{\partial v_2} + \frac{1}{2} \sigma_1^2 e^{2v_{1t}} \frac{\partial^2 F}{\partial X^2} + \rho_{12} \sigma_1^2 e^{2v_{1t}} \frac{\partial^2 F}{\partial X \partial v_1} + \frac{1}{2} \sigma_2^2 e^{2v_{2t}} \frac{\partial^2 F}{\partial v_2^2} + \rho_{13} \sigma_1 \sigma_2 e^{2v_{1t}} \frac{\partial^2 F}{\partial X \partial v_2} + \rho_{23} \sigma_2 \sigma_3 e^{2v_{2t}} \frac{\partial^2 F}{\partial v_1 \partial v_2} + \frac{1}{2} \sigma_3^2 e^{2v_{3t}} \frac{\partial^2 F}{\partial v_1^2} + \frac{1}{2} \sigma_3^2 e^{2v_{3t}} \frac{\partial^2 F}{\partial v_2^2} + \frac{1}{2} \sigma_3^2 e^{2v_{3t}} \frac{\partial^2 F}{\partial v_3^2} = 0
$$

subject to the terminal condition $F(X_T, v_{1T}, v_{2T}, T) = S(T) = e^{h_T + X_T}$.
The solution now is of the form $F_T^T = e^{h_T + G(t, T) + B(t, T)X_T + C(t, T)v_T + D(t, T)v_{2T}}$ and (A5) yields the system of ordinary differential equations

$$B'' - \kappa B = 0$$  \hspace{1cm} (A6)

$$C'' + \kappa B - a_1 C = 0$$  \hspace{1cm} (A7)

$$D' + \frac{\sigma_2^2}{2} D^2 - D\left(a_2 - \rho_{13} \sigma_2 e^{\sigma_1} B - \rho_{23} \sigma_1 e^{\sigma_2} C\right) + \frac{1}{2} e^{2\sigma_1} B^2 + \rho_{13} \sigma_1 B C e^{2\sigma_2} + \frac{1}{2} \sigma_1^2 e^{2\sigma_1} C^2 = 0$$  \hspace{1cm} (A8)

$$G + a_1 b_1 C + a_2 b_2 D = 0$$  \hspace{1cm} (A9)

with initial conditions $G(T, T) = 0$, $B(T, T) = 1$, $C(T, T) = 0$, $D(T, T) = 0$. Equations (A6) and (A7) are the same as Equations (A2) and (A3), while Equation (A8) is the same as Equation (25). Integrating Equation (A9) provides Equation (24).

### Appendix 2

**Kalman filter estimation of the two models.** The models presented in Sections 4 and 5 were established in a continuous-time framework, while the Kalman filter operates in discrete time. To describe the method, we need some additional notation: if $t_j$ is the observation date ($j=1,…, J$) denote $Y_j = [\ln F^T_{t_j}]$, the $N \times 1$ vector of futures log prices observed at date $t_j$ and $\Delta_j = t_{j+1} - t_j$ the time increment. We reserve the subscript $j$ for the discrete version, which is equivalent to $t_j$ in the continuous-time version.

#### A. The two-factor model

In this model, both state variables (the stochastic component of the spot price and its short-term mean) are not observable and are estimated through the Kalman filter. Denote the state vector at time $t$ as $Z_t = [X_t, v_t]'$.

The measurement equation relates the observable vector $Y_j$ to the state vector $Z_j$ and is defined here by

$$Y_j = M_j + L_j Z_j + \nu_j, \quad j = 1,\ldots,J$$  \hspace{1cm} (A10)

where

$$M_j = [h_{t_j} + A(t_j, T_j)], \quad i=1,\ldots, N, \quad N \times 1 \text{ vector}$$

$$L_j = [B(t_j, T_j), C(t_j, T_j)], \quad i=1,\ldots, N, \quad N \times 2 \text{ vector}$$

$\nu_j$ is a $N \times 1$ vector of serially uncorrelated disturbances with $E(\nu_j) = 0$, $\text{Var}(\nu_j) = \Omega$

where $\Omega$ is a diagonal matrix with $\Omega_{ii} = \omega_i$ ($i=1,\ldots, N$).
The \( \nu_j \) are called “measurement errors”; they take into account bid-ask spreads, price limits, nonsimultaneity of the observations, errors in the data, etc…

Equations (4) and (6) can be rewritten as the transition equation

\[
dZ_t = (U_t + HZ_t)dt + V_t dW_t
\]  

(A11)

where

\[
U_t = \left[ \lambda X e^{\nu_t}, (a_t b_1 + \lambda_1 \xi_1 e^{\nu_t}) \right]'; \quad H = \begin{bmatrix} -\kappa & \kappa \\ 0 & -a_t \end{bmatrix};
\]

\[
V_t \text{ is such that } V_t V_t' = e^{2\nu_t} \begin{bmatrix} \sigma^2 & \rho_{12} \sigma_1 \xi_1 \\ \rho_{12} \sigma_1 \xi_1 & \xi_1^2 \end{bmatrix}
\]

\( W \) is a standard Brownian motion in \( R^2 \).

Consider a subinterval \( [t_j, t_{j+1}] \) and approximate the vector \( U_t \) and the matrix \( V_t \) over this subinterval by their values at \( t_j \). Then the transition equation (A11) is approximated by

\[
dZ_t = (U_j + HZ_j)dt + V_j dW_t
\]  

(A12)

where

\( U_j = U_{t_j}; \quad V_j = V_{t_j} \)

The linear stochastic differential equation (A12) can be integrated to yield

\[
Z_{j+1} = e^{H\Delta_j} Z_j + \int_0^{\Delta_j} e^{H(\Delta_j-u)} U_j du + \int_0^{\Delta_j} e^{H(\Delta_j-u)} V_j dW_u
\]

where \( Z_j = Z_{t_j} \)

or equivalently,

\[
Z_{j+1} = e^{H\Delta_j} Z_j + G_{12}(\Delta_j) + \tilde{V}_j \varepsilon_j
\]  

(A13)

where \( \varepsilon_j \) is a \( 2 \times 1 \) vector of serially uncorrelated disturbances with

\( E(\varepsilon_j) = 0 \) and \( Var(\varepsilon_j) = I_{2 \times 2} \) (identity matrix of size \( 2 \times 2 \))

\( \tilde{V}_j \) is such that \( \tilde{V}_j \tilde{V}_j' = R_{12}(\Delta_j)R_{11}(\Delta_j)' \)

and the sub-matrices \( G_{12}(\Delta_j), R_{11}(\Delta_j) \) and \( R_{12}(\Delta_j) \) are determined by

\[
\exp \left[ \begin{bmatrix} H & U_{j-1} \\ 0 & 0 \end{bmatrix} \Delta_j \right] = \begin{bmatrix} G_{11}(\Delta_j) & G_{12}(\Delta_j) \\ 0 & G_{22}(\Delta_j) \end{bmatrix}
\]
and
\[
\exp\left(\begin{bmatrix}
H & \vec{V}_{j-1}\vec{V}_{j-1}'
\end{bmatrix}\Delta_j\right) = \begin{bmatrix}
R_{11}(\Delta_j) & R_{12}(\Delta_j) \\
0 & R_{22}(\Delta_j)
\end{bmatrix}
\]

(Moreover, it can be shown in particular that
\[
R_{11}(\Delta_j) = e^{H\Delta_j}, R_{22}(\Delta_j) = e^{-H\Delta_j}, G_{12}(\Delta_j) = \int_0^{\Delta_j} e^{H(\Delta_j-u)}U_j du.
\]

Equation (A13) is the approximate discrete-time version of the transition equation (A11).

Let \( \hat{Z}_{j|j} \) denote the optimal estimator of \( Z_j \) based on the observations up to and including \( Y_j^F \). Let \( P_{j|j} \) denote the covariance matrix of the estimation error:
\[
P_{j|j} = E\left(\left(\begin{bmatrix}
Z_j - \hat{Z}_{j|j} \\
\hat{Z}_{j|j} - \hat{Z}_{j|j}
\end{bmatrix}\right)\left(\begin{bmatrix}
Z_j - \hat{Z}_{j|j} \\
\hat{Z}_{j|j} - \hat{Z}_{j|j}
\end{bmatrix}\right)\right)
\]

where we define \( \hat{Z}_{0|0} = Z_0 \) and \( P_{0|0} = P_0 \). Equation (A13) implies that given \( \hat{Z}_{j|j} \) and \( P_{j|j} \), the optimal forecast of \( Z_{j+1} \) and the associated covariance matrix of the estimation error are
\[
\hat{Z}_{j+1|j} = e^{H\Delta_j} \hat{Z}_{j|j} + G_{12}(\Delta_j) \\
P_{j+1|j} = e^{H\Delta_j} P_{j|j} e^{H\Delta_j} + \vec{V}_j \vec{V}_j'
\]

Equations (A14) and (A15) are known as the prediction equations.

Once a new observation \( Y_{j+1}^F \) becomes available, the estimator \( \hat{Z}_{j+1|j} \) can be updated. The updating equations are:
\[
\hat{Z}_{j+1|j+1} = \hat{Z}_{j+1|j} + K_{j+1} \hat{u}_{j+1} \\
\hat{P}_{j+1|j+1} = \hat{P}_{j+1|j} - P_{j+1|j} L_{j+1} R_{j+1}^{-1} R_{j+1} L_{j+1} P_{j+1|j}
\]

where
\[
K_{j+1} = P_{j+1|j} L_{j+1} R_{j+1}^{-1} \\
R_{j+1} = L_{j+1} P_{j+1|j} L_{j+1}' + \Omega \\
\hat{u}_{j+1} = Y_{j+1}^F - \left( M_{j+1} + L_{j+1} \hat{Z}_{j+1|j} \right)
\]

Conditional on \( Y_1^F, ..., Y_j^F, Z_{j+1} \) is normally distributed with a mean of \( \hat{Z}_{j+1|j} \) and a covariance matrix of \( P_{j+1|j} \). From Equations (A10) and (A20),
\[
\hat{u}_{j+1} = L_{j+1}(Z_{j+1} - \hat{Z}_{j+1|j}) + \nu_{j+1}
\]

Equation (A21) implies that the conditional distribution of \( \hat{u}_{j+1} \) is normal with zero mean and a covariance matrix of \( R_{j+1} \). Therefore, the log-likelihood function can be written as:
The parameter estimates can be obtained by maximizing (A22). It is clear from Equations (A14) to (A17) that the log-likelihood function value depends on $Z_0$ and $P_0$. Rosenberg’s algorithm consists in setting $P_0 = 0$ and finding the maximum likelihood estimator of $Z_0$, conditional on the other parameters in the model. By this way, $Z_0$ is concentrated out of the log-likelihood function.

B. The three-factor model

We want however to mention that in order to overcome the problem of missing data in the monthly inventory observations, we approached the three-state variable model in two ways:

a) by creating estimates of these data through linear interpolation of observed ones

b) by treating the monthly scarcity as another state variable for which the measurements equation has the simple form

$$s(t) = \hat{s}(t) + \varepsilon_t$$

where $\hat{s}(t)$ is the expected value of $S(t)$ accounting for the quarterly pattern.

The numbers obtained through the Kalman filter for the parameters under the two procedures are totally indistinguishable, which is not surprising given the linearity exhibited by the inventory behavior within a quarter (see Figure 1C). Hence, we discuss below the approach a).

In this model, only the stochastic component of the spot price and its short-term mean are unobservable; the scarcity variable is directly obtained as the inverse of the inventory numbers. Denote the state vector at time $t$ as $Z_t = [X_t, v_{s_t}]'$.

The measurement equation relating the observable vector $Y_{t}^F$ to the state vector $Z_t$ is given by

$$Y_{t}^F = M_j + L_j Z_j + \omega_{j}, j=1,\ldots,J$$

(A23)

where

$$M_j = [h_{t_j} + G(t_j, T_i) + D(t_j, T_i) v_{2t_j}], \ i=1,\ldots, N, N \times 1 \text{ vector}$$

$$L_j = [B(t_j, T_i), C(t_j, T_i)], \ i=1,\ldots, N, N \times 2 \text{ vector}$$

$\omega_{j}$ is a $N \times 1$ vector of serially uncorrelated disturbances with $E(\omega_{j}) = 0, \ Var(\omega_{j}) = \Omega$

where $\Omega$ is a diagonal matrix.

Equations (15) and (16) can be rewritten as

$$dZ_t = (U_t + HZ_t)dt + V_t dW_t$$

(A24)

where

$$U_t = [\lambda_x e^{v_{1}}, v_{2t}, (a_i b_1 + \lambda_i \sigma_1 e^{v_{1}}, v_{2t})]'$$

$$H = \begin{bmatrix} -\kappa & \kappa \\ 0 & -a_1 \end{bmatrix}$$
$V_t$ is such that $V_t' = e^{2\rho_1} \left[ \begin{array}{cc} 1 & \rho_{12} \sigma_1 \\ \rho_{12} \sigma_1 & \sigma_1^2 \end{array} \right] V_{2t}$.

$W$ is a standard Brownian motion in $\mathbb{R}^2$.

Using the same approximation method as in the two-factor model, we obtain the following discrete-time transition equation:

$$Z_j = e^{th_j} Z_{j-1} + G_{12}(\Delta_j) + V_j e_j$$

(A25)

where $e_j$ is a $2 \times 1$ vector of serially uncorrelated disturbances with $E(e_j) = 0$ and $Var(e_j) = I_d$.

$V_j$ is such that $V_j' V_j = R_{12}(\Delta_j) R_{11}(\Delta_j)$

and the sub-matrices $G_{12}(\Delta_j), R_{11}(\Delta_j)$ and $R_{12}(\Delta_j)$ are computed from the matrix exponentials defined in the same manner as in the two-factor model.

The Kalman filter and Rosenberg’s algorithm to find $Z_0$ then can be applied to the measurement equation (A23) and the transition equation (A25) as it has been shown for the two-factor model.