Online Companion for

"Maximizing Revenue of Perishable Assets with A Risk Factor"

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Youyi Feng
National University of Singapore
Republic of Singapore

Baichun Xiao
Seton Hall University, New Jersey
Long Island University, C.W. Post Campus

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On-Line Appendix

Proof of Lemma 1

By the definition of Δ_j^i, we derive

$$\begin{aligned}
E[\Delta_j^i|\mathcal{F}_{t+jh}^i] &= p_i E\left[\int_{t+jh}^{t+(j+1)h} 1_{\{N_i(s)-N_i(t)<n\}}\lambda_i ds \middle| \mathcal{F}_{t+jh}^i\right] \\
&= p_i E\left[\int_{t+jh}^{t+(j+1)h} 1_{\{N_i(s)-N_i(t+jh)<n-N_i(t+jh)+N_i(t)\}}\lambda_i ds \middle| \mathcal{F}_{t+jh}^i\right] \\
&= p_i \int_{t+jh}^{t+(j+1)h} \Sigma_{k\geq 0} 1_{\{k<n-N_i(t+jh)+N_i(t)\}} \frac{[\lambda_i(s-t-jh)]^k}{k!} e^{-\lambda_i(s-t-jh)}\lambda_i ds \\
&= \lambda_i p_i\left[\int_{t+jh}^{t+(j+1)h} 1_{\{0<n-N_i(t+jh)+N_i(t)\}} e^{-\lambda_i(s-t-jh)} ds\right] + o(h^2)
\end{aligned}$$

The difference between the actual and expected sales revenue, thus, is given by

$$\begin{aligned}
\Delta_j^i - E[\Delta_j^i|\mathcal{F}_{t+jh}^i] &= p_i(N_i(t+(j+1)h)-N_i(t))\wedge n - p_i(N_i(t+jh)-N_i(t))\wedge n \\
&\quad -\lambda_i p_i\left[\int_{t+jh}^{t+(j+1)h} 1_{\{0<n-N_i(t+jh)+N_i(t)\}} e^{-\lambda_i(s-t-jh)} ds\right] + o(h^2),
\end{aligned}$$

Take a summation over all intervals within $[t,\ u]$ and replace $t+jh$ by $[s]$ for all $s \in [t+jh, t+(j+1)h)$. As $[s] \to s$ when $h \to 0$, we have

$$\Sigma_j(\Delta_j^i - E[\Delta_j^i|\mathcal{F}_{t+jh}^i]) \to p_i(N_i(u)-N_i(t))\wedge n - \lambda_i p_i \int_t^u 1_{\{0<n-N_i(s)+N_i(t)\}} ds \tag{22}$$

almost surely. As $\Sigma_j(\Delta_j^i - E[\Delta_j^i|\mathcal{F}_{t+jh}^i])$ is bounded above, applying the dominant convergence theorem leads to

$$E\left[\Sigma_j\left(\Delta_j^i - E(\Delta_j^i|\mathcal{F}_{t+jh}^i)\right)\right]^2 \to E\left[p_i((N_i(u)-N_i(t))\wedge n) - \lambda_i p_i \int_t^u 1_{\{N_i(s)-N_i(t)<n\}} ds\right]^2,$$

as $h \to 0$. Following Karr (1986), we can show that

$$E\left[p_i((N_i(u)-N_i(t))\wedge n) - \lambda_i p_i \int_t^u 1_{\{N_i(s)-N_i(t)<n\}} ds\right]^2 = R_i(t,u,n).$$

□

Proof of Lemma 2

$n=0$ is trivial, so assume $n \geq 1$. Observe that

$$E[(N_i(T)-N_i(t))\wedge n] = \sum_{k=1}^n P[N_i(T)-N_i(t)\geq k].$$

Hence

$$f(t,n) = p_2 \sum_{k=1}^n P[N_2(T)-N_2(t)\geq k].$$

It can also be shown (we omit the detail) that

$$R_i(t,T,n) = p_i^2 \sum_{k=1}^{n} P[N_i(T) - N_i(t) \geq k].$$

The conclusion then follows immediately. □

Proof of Lemma 3

It suffices to verify (ii). Since $f(t,n)$ is uniformly bounded ($\leq Mp_2$) and continuously differentiable with respect to t over $[0,\, T]$, $f(t,n)$ is in the domain of generator $\mathcal{G}$ and so is $\pi(t,n)$. Notice for $n \geq 1$,

$$\frac{\partial h_i(t,n)}{\partial t} = -\lambda_i P[N_i(T) - N_i(t) \leq n-1].$$

Hence,

$$\begin{aligned}
\mathcal{G}\pi(t,n) &= \frac{\partial \pi(t,n)}{\partial t} + \lambda_1[\pi(t,n-1) - \pi(t,n)] \\
&= -\tilde{p}_2\lambda_2 P[N_2(T) - N_2(t) \leq n-1] - c\lambda_1 p_1^2 P[N_1(T) - N_1(t) \leq n-1] \\
&\quad -\lambda_1\tilde{p}_2 P[N_2(T) - N_2(t) \geq n] - c\lambda_1 p_1^2 P[N_1(T) - N_1(t) \geq n] \\
&\quad -\lambda_2\tilde{p}_2[1 - P(N_2(T) - N_2(t) \geq n)] - \lambda_1\tilde{p}_2[1 - P(N_2(T) - N_2(t) \geq n)] - c\lambda_1 p_1^2 \\
&= \tilde{r}_1 - \tilde{r}_2 + \tilde{p}_2(\lambda_2 - \lambda_1) P[N_2(T) - N_2(t) \geq n] - \lambda_1 p_1.
\end{aligned}$$

□

Proof of Theorem 1

Since $\bar{V}(t,n)$ is defined over a finite horizon and continuous, it is in the domain of the infinitesimal generator $\mathcal{G}$. For $s \geq t$, let

$$\begin{aligned}
m(s) &= \bar{V}(s, (n - N_1(s) + N_1(t))^+) - \bar{V}(t,n) \\
&\quad - \int_t^s \mathcal{G}\bar{V}(u, (n - N_1(u) + N_1(t))^+ du.
\end{aligned}$$

It is clear that $m(s)$ is a martingale and $E[m(s)|\mathcal{F}_t] = 0$. By the optional sampling theorem, for $\tau \geq t$ we have

$$E[\bar{V}(\tau, (n - N_1(\tau) + N_1(t))^+)] - E\int_t^\tau [\mathcal{G}\bar{V}(u, (n - N_1(u) + N_1(t))^+)]du = \bar{V}(t,n),$$

which is equivalent to

$$\begin{aligned}
&E[\bar{V}(\tau, (n - N_1(\tau) + N_1(t))^+)] \\
&-E\int_t^\tau [\mathcal{G}(\bar{V} + \pi)(u, (n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u) - N_1(t) < n\}}]du \\
&= \bar{V}(t,n) - E\int_t^\tau [\mathcal{G}\pi(u, (n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u) - N_1(t) < n\}}]du. \qquad (23)
\end{aligned}$$

In line with conditions (iii) and (iv), the left side of (23) is nonnegative. Thus, for an arbitrary stopping time $\tau \in [t,T]$,

$$\bar{V}(t,n) \geq E\int_t^\tau [\mathcal{G}\pi(u, (n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u) - N_1(t) < n\}}]du.$$

Taking supremum over all those stopping times $t \le \tau \le T$ results in

$$\bar{V}(t,n) \ge \tilde{V}(t,n). \tag{24}$$

To prove the converse inequality, we define a specific stopping time

$$\sigma = \inf\{t \le s \le T : \ \bar{V}(s,(n - N_1(s) + N_1(t))^+) = 0\}.$$

Clearly such a σ is well defined since $\bar{V}(T,n) = 0$ $(0 \le n \le M)$. Substituting σ into the martingale $m(s)$ and applying the optional sampling theorem again results in

$$\begin{aligned}
&E[\bar{V}(\sigma,(n - N_1(\sigma) + N_1(t))^+)] \\
&-E\int_t^\sigma [\mathcal{G}(\bar{V}+\pi)(u,(n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u)-N_1(t)<n\}}]du \\
&= \bar{V}(t,n) - E\int_t^\sigma [\mathcal{G}\pi(u,(n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u)-N_1(t)<n\}}]du.
\end{aligned} \tag{25}$$

The definition of σ and (iv) implies $E[\bar{V}(\sigma,(n - N_1(\sigma) + N_1(t))^+)] = 0$ and $\mathcal{G}(\bar{V}+\pi)(u,(n - N_1(u) + N_1(t))^+) + \lambda_1 p_1 = 0$ for all $u \in [t,\sigma]$. Consequently, the left hand side of the above equation is zero, giving that

$$\bar{V}(t,n) = E\int_t^\sigma [\mathcal{G}\pi(u,(n - N_1(u) + N_1(t))^+)) + \lambda_1 p_1 \mathrm{l}_{\{N_1(u)-N_1(t)<n\}}]du \le \tilde{V}(t,n). \tag{26}$$

Coupling (24) and (26) leads to the conclusion. □

Proof of Theorem 2

We show the assertion through induction. When $n = 1$, $L(t,1) = \mathcal{G}\pi(t,1) + \lambda_1 p_1$ is an increasing function of t. Hence, for $t \le x_1$, we have $\bar{V}(t,1) = 0$ and

$$L(t,1) = \mathcal{G}\pi(t,1) + \lambda_1 p_1 \le \mathcal{G}\pi(x_1,1) + \lambda_1 p_1. \tag{27}$$

We claim $L(t,1) \le 0$ since otherwise the definition of x_1 is violated. As a result,

$$\begin{aligned}
\Psi(t,1) &= \int_t^T L(s,1)e^{-\lambda_1(s-t)}ds \\
&= \bar{V}(x_1,1) + \int_t^{x_1} L(s,1)e^{-\lambda_1(s-t)}ds \\
&= \int_t^{x_1} L(s,1)e^{-\lambda_1(s-t)}ds \le 0.
\end{aligned}$$

When $t > x_1$, (12) leads to

$$\mathcal{G}(\bar{V}+\pi)(t,1) + \lambda_1 p_1 = \frac{\partial \bar{V}(t,1)}{\partial t} - \lambda_1 \bar{V}(t,1) + \mathcal{G}\pi(t,1) + \lambda_1 p_1.$$

It can be shown from (12) that

$$\frac{\partial \bar{V}(t,1)}{\partial t} = \lambda_1 \bar{V}(t,1) - \mathcal{G}\pi(t,1) - \lambda_1 p_1,$$

and $\mathcal{G}(\bar{V}+\pi)(t,1) + \lambda_1 p_1 = 0$ follows. Hence, $\bar{V}(t,1)$ satisfies conditions (i) through (iv) in Theorem 1. In addition, we have $\bar{V}(t,1) \ge \bar{V}(t,0) = 0$.

Assume the statement holds for $n \leq k < M$; i.e., there exist k time thresholds with $T \geq x_1 \geq \cdots \geq x_k \geq 0$ such that $\bar{V}(t,n)$ $(1 \leq n \leq k)$ derived from (12) satisfy conditions (i) through (iv) in Theorem 1 and the inequality

$$\bar{V}(t,n) \geq \bar{V}(t,n-1), \qquad n = 1, \cdots, k.$$

Hence,

$$L(t,k+1) \geq L(t,k)$$

since $\mathcal{G}\pi(t,k+1) \geq \mathcal{G}\pi(t,k)$, and by induction, $\bar{V}(t,k) \geq \bar{V}(t,k-1)$. This inequality implies for $0 \leq t \leq T$,

$$\Psi(t,k+1) = \int_t^T L(t,k+1)e^{-\lambda_1(s-t)}ds \geq \int_t^T L(t,k)e^{-\lambda_1(s-t)}ds = \Psi(t,k)$$

and immediately, $x_k \geq x_{k+1}$. Putting together these facts results in

$$\bar{V}(t,k+1) \geq \bar{V}(t,k).$$

To show $\bar{V}(t,k+1)$ satisfies (i) - (iv) of Theorem 1. For $t < x_{k+1}$, $\bar{V}(t,k+1) = 0$ and $\frac{\partial \bar{V}(t,k+1)}{\partial t} = 0$. Hence,

$$\begin{aligned}
& \mathcal{G}(\bar{V}+\pi)(t,k+1) + \lambda_1 p_1 \\
= \; & \frac{\partial \bar{V}(t,k+1)}{\partial t} + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & \lambda_1 \bar{V}(t,k) + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & L(t,k+1) \\
< \; & L(x_{k+1},k+1) \\
\leq \; & 0.
\end{aligned}$$

When $t > x_{k+1}$, $\bar{V}(t,k+1)$ is defined by (12) and

$$\begin{aligned}
& \mathcal{G}(\bar{V}+\pi)(t,k+1) + \lambda_1 p_1 \\
= \; & \frac{\partial \bar{V}(t,k+1)}{\partial t} + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & -L(t,k+1) + \lambda_1 \bar{V}(t,k+1) + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & 0.
\end{aligned}$$

Thus, $\bar{V}(t,k+1)$ meets (i) - (iv). This completes the proof. □

Proof of Theorem 3

The monotonicity of x_n can be easily seen in the proof of Theorem 2, so it suffices to prove the second part of the theorem. Without loss of generality, let $0 < x_{n+1} \leq x_n < T$. If $x_{n+1} = x_n$, then from the proof of Theorem 2, we have $L(t,n+1) = L(t,n)$ and $\bar{V}(t,n+1) = \bar{V}(t,n)$ for any $t > x_n$ (or x_{n+1}). This will result

$$\begin{aligned}
& L(t,n+1) - L(t,n) \\
= \; & [\mathcal{G}\pi(t,n+1) + \lambda_1 p_1 + \lambda_1 \bar{V}(t,n)] - [\mathcal{G}\pi(t,n) + \lambda_1 p_1 + \lambda_1 \bar{V}(t,n-1)] \\
\geq \; & [\mathcal{G}\pi(t,n+1) - \mathcal{G}\pi(t,n)] + \lambda_1[\bar{V}(t,n) - \bar{V}(t,n-1)] \\
= \; & 0.
\end{aligned}$$

Since $\bar{V}(t,n) - \bar{V}(t,n-1) \geq 0$, and

$$\mathcal{G}\pi(t,n+1) - \mathcal{G}\pi(t,n) \geq 0.$$

with equality being true only when $t = T$. This leads to a contradiction. □

Proof of Corollary 2 It is easy to see that $\bar{V}(t,1) > 0$, $0 \leq t < T$, and $x_1 = 0$. By induction, we have

$$L(t,n) > 0, \quad 0 \leq t < T, \quad 1 \leq n \leq M,$$

which implies

$$\bar{V}(t,n) > 0, \quad 0 \leq t < T; \quad x_n = 0, \ 1 \leq n \leq M.$$

□

Proof of Corollary 3

Since $c \in [(p_1+p_2)^{-1}, p_2^{-1})$, it implies that $\tilde{p}_2 > 0$. Notice that due to $\lambda_1 > \lambda_2$,

$$\begin{aligned}
\mathcal{G}\pi(t,n) + \lambda_1 p_1 &= \tilde{r}_1 - \tilde{r}_2 + \tilde{p}_2(\lambda_2 - \lambda_1) P[N_2(T) - N_2(t) \geq n] \\
&\geq \tilde{r}_1 - \tilde{r}_2 + \tilde{p}_2(\lambda_2 - \lambda_1) \\
&= \lambda_1(p_1 - p_2)[1 - c(p_1+p_2)] \\
&\geq 0.
\end{aligned}$$

The last inequality becomes equality if and only if $n = 0$ and $c = (p_1+p_2)^{-1}$. Thus,

$$\mathcal{G}\pi(t,n) + \lambda_1 p_1 > 0.$$

The conclusion then follows from Corollary 2. □

The following preliminaries are necessary for understanding the proofs of Theorem 4 and 7.

A kernel $K(t,s)$ of two real variables t and s over linearly ordered one dimensional sets A and B, respectively, is said to be *totally positive of orders n*, denoted by TP_n, if for all $t_1 < t_2 < \cdots < t_r$ in A and all $s_1 < s_2 < \cdots < s_r$ in B $(r \leq n)$, we have

$$\det \begin{vmatrix} K(t_1,s_1) & \cdots & K(t_1,s_\pi) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_r,s_1) & \cdots & K(t_r,s_\pi) \end{vmatrix} \geq 0.$$

A kernel $K(t,s)$ of the same type is said to be *sign-regular of orders n*, denoted by SR_n, if it is sign-consistent of every order from 1 through n; i.e., if there exists a sequence of constants $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ with alternate signs $+$ and $-$ such that for each $r = 1, \cdots, n$,

$$\epsilon_r \det \begin{vmatrix} K(t_1,s_1) & \cdots & K(t_1,s_\pi) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ K(t_r,s_1) & \cdots & K(t_r,s_\pi) \end{vmatrix} \geq 0.$$

Let ϕ be a Borel measurable function on B and let $\psi(t) = \int_B \phi(s)K(t,s)ds$ be finite for each t in A. The *variation diminishing* property states that if $K(t,s)$ is TP_n and ϕ has at most $n-1$ changes in sign, then ψ has at most $n-1$ changes in sign. Moreover, if ψ and ϕ have the same number of changes in sign, then they exhibit the same sequence of signs when the respective arguments traverse the domains from left to right. Let $K(t,s) = e^{-\lambda_1(s-t)}1_{\{s\geq t\}}$ and $A = B = [0,T]$. It can be verified that $K(t,s)$ is TP_n for all n. The lemma below is the restatement of Proposition 1.3.1 in Karlin (1968). To verify $\bar{V}(t,n)$ satisfy the conditions described in Theorem 1, we need following lemmas.

Lemma 4 *Suppose $\int_B K(x,y)d\mu(y) = 1$ for all $x \in A$. If $K(\cdot,\cdot)$ is SR_2 and $f(y)$ is monotone, then $g(x) = \int_B K(x,y)f(y)d\mu(y)$ is monotone.*

According to the notion of Karlin, TP_n implies SR_n. Applying this fact to the kernel $K(x,y) = e^{-\lambda_1(y-x)}1_{\{y\geq x\}}$ leads to a decreasing property of $\bar{V}(t,n)$ in t.

Lemma 5 *For $n \geq 0$, the optimal premium $\bar{V}(t,n)$ is decreasing in t.*

Proof. We justify the assertion by induction. Let $B = [0,\infty)$ and $A = [0,T]$. Through a simple calculation we shall derive

$$\int_B K(t,s)ds = \frac{1}{\lambda_1}, \quad \forall t \in \mathrm{A}.$$

By the definition of y_1, $\mathcal{G}\pi(y,1)1_{\{y\leq y_1\}}$ is decreasing. Following Lemma 5, for any $t \in A = [0,T]$,

$$\bar{V}(t,1) = \int_B K(t,y)[\mathcal{G}\pi(y,1) + \lambda_1 p_1]1_{\{y\leq y_1\}}dy$$

is decreasing. So the conclusion is correct for $n = 1$. Assume that $\bar{V}(t,n)$ is decreasing. This implies that $L(t,n) = \mathcal{G}f(t,n) + \lambda_1 p_1 + \lambda_1 \bar{V}(t,n-1)$ is decreasing, and so is $L(y,n)1_{\{y\leq y_{n-1}\}}$. Applying Lemma 5 again, we see

$$\bar{V}(t,n) = \int_B K(t,n)L(y,n)1_{\{y\leq y_{n-1}\}}dy$$

is decreasing in t. □

Proof of Theorem 4

The proof is done by induction. For $n = 1$ and $t < y_1 \leq T$, as $L(t,1) = \mathcal{G}\pi(t,1) + \lambda_1 p_1$ is a decreasing function of t,

$$\bar{V}(t,1) = \int_t^{y_1} L(s,1)e^{-\lambda_1(s-t)}ds > 0.$$

The construction of $\bar{V}(t,1)$ ensures

$$\mathcal{G}[\bar{V} + \pi](t,1) + \lambda_1 p_1 = \frac{\partial \bar{V}(t,1)}{\partial t} - \lambda_1 \bar{V}(t,1) + L(t,1) = 0.$$

When $t > y_1$, $\bar{V}(t,1) = 0$ and $\frac{\partial \bar{V}(t,1)}{\partial t} = 0$. This gives

$$\begin{aligned} & \mathcal{G}(\bar{V} + \pi)(t,1) + \lambda_1 p_1 \\ = \quad & \frac{\partial \bar{V}(t,1)}{\partial t} + \lambda_1[\bar{V}(t,0) - \bar{V}(t,1)] + \mathcal{G}\pi(t,1) + \lambda_1 p_1 \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{G}\pi(t,1) + \lambda_1 p_1 \\
&\leq \mathcal{G}\pi(y_1,1) + \lambda_1 p_1 \\
&= 0.
\end{aligned}
$$

Hence, $\bar{V}(t,1)$ meets (i) through (iv) and is equal to $\tilde{V}(t,1)$.

Suppose the conclusion remains true for $n \leq k < M$; i.e., there exists a decreasing set of k thresholds $T \geq y_1 \geq \cdots \geq y_k \geq 0$ such that $\bar{V}(t,n) = \tilde{V}(t,n)$ $(1 \leq n \leq k)$. Let $t > y_k$, then $\bar{V}(t,k) = 0$ and

$$
\begin{aligned}
L(t,k+1) &= \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 + \lambda_1 \bar{V}(t,k) \\
&\leq \mathcal{G}\pi(t,k) + \lambda_1 p_1 \\
&\leq L(t,k) \\
&\leq 0,
\end{aligned}
$$

which implies $y_k \geq y_{k+1}$. For $t < y_{k+1}$, $\bar{V}(t,k+1)$ is given by (16) and is positive due to the monotonicity stated by Lemma 5. Moreover,

$$
\begin{aligned}
& \mathcal{G}(\bar{V}+\pi)(t,k+1) + \lambda_1 p_1 \\
= \; & \frac{\partial \bar{V}(t,k+1)}{\partial t} + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & -L(t,k+1) + \lambda_1 \bar{V}(t,k+1) + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & 0,
\end{aligned}
$$

with the boundary condition $\bar{V}(y_{k+1},k+1) = 0$. If $t > 0$, then $\bar{V}(t,k+1) = 0$ and

$$
\begin{aligned}
& \mathcal{G}(\bar{V}+\pi)(t,k+1) + \lambda_1 p_1 \\
= \; & \frac{\partial \bar{V}(t,k+1)}{\partial t} + \lambda_1[\bar{V}(t,k) - \bar{V}(t,k+1)] + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & \lambda_1 \bar{V}(t,k) + \mathcal{G}\pi(t,k+1) + \lambda_1 p_1 \\
= \; & L(t,k+1) \\
\leq \; & 0.
\end{aligned}
$$

Thus, $\bar{V}(t,k+1)$ satisfy (i) - (iv) of Theorem 1, and the proof is completed. □

Proof of Theorem 5

The theorem is a special case of Theorem 3 in Feng and Xiao (1996). □

Other Results for the Markdown Solutions

Theorem 7 *The sequence $\mathcal{Y} = \{y_n : n = 1, \ldots, M\}$ is decreasing; i.e., $y_1 \geq y_2 \geq \cdot \geq y_M$. In particular, if for some n_1 and n_2, $1 \leq n_1 < n_2 \leq M$, $y_{n_2} > 0$ and $y_{n_1} < T$, then y_n is strictly decreasing for any $n_1 \leq n \leq n_2$.*

Proof. The general monotone property of y_n is shown in the proof of Theorem 4. The proof of strict monotonicity is similar to that of Theorem 3 and is omitted. □

Corollary 6 *If for all* $0 \le t \le T, \quad 1 \le n \le M$,

$$\mathcal{G}\pi(t,n) + \lambda_1 p_1 \le 0,$$

then $\bar{V}(t,n) = 0$ *and* $y_1 = \cdots = y_M = 0$.

Proof. In view of Theorem 4, it is clear that $\bar{V}(t,1) = 0$. Suppose the statement is true for all $n < k$, then

$$\mathcal{G}\pi(t,k) + \lambda_1 p_1 + \lambda_1 \bar{V}(t,k-1) = \mathcal{G}\pi(t,n) + \lambda_1 p_1 \le 0.$$

Applying Theorem 4 leads to $\bar{V}(t,n) = 0$. □

Corollary 7 *If* $c \in [(p_1+p_2)^{-1}, p_2^{-1})$, *then*

$$\mathcal{G}\pi(t,n) + \lambda_1 p_1 \le 0$$

holds for all $1 \le n \le M$. *As a result, optimal premium* $\bar{V}(t,n)$ *are zero for* $1 \le n \le M$, *and* $y_1 = \cdots = y_M = 0$.

Proof. Since $\lambda_1 < \lambda_2$ and $\tilde{p}_2 > 0$, it ensures

$$\mathcal{G}\pi(t,n) + \lambda_1 p_1 \le \tilde{r}_1 - \tilde{r}_2 + \tilde{p}_2(\lambda_2 - \lambda_1) = \lambda_1(p_1 - p_2)[1 - c(p_1+p_2)] \le 0.$$

□

Corollary 8 *In the markdown situation, a necessary condition for the higher price to be enacted in the markdown case is* $c < (p_1+p_2)^{-1}$. *If the risk factor* $c > (p_1+p_2)^{-1}$, *then keeping the higher price is optimal throughout the entire sales horizon.*

Proof of Theorem 6

Let $c_1 < c_2 < (p_1+p_2)^{-1}$, we claim that

$$\mathcal{G}\pi^{c_1}(t,n) + \lambda_1 p_1 < \mathcal{G}\pi^{c_2}(t,n) + \lambda_1 p_1, \tag{28}$$

because

$$\begin{aligned}
& [\mathcal{G}\pi^{c_1}(t,n) + \lambda_1 p_1] - [\mathcal{G}\pi^{c_2}(t,n) + \lambda_1 p_1] \\
= \;& (c_2 - c_1)\left\{(\lambda_1 p_1^2 - \lambda_2 p_2^2) + p_2^2(\lambda_2 - \lambda_1)P[N_2(T) - N_2(t) \ge n]\right\} \\
\le \;& (c_2 - c_1)(\lambda_1 p_1^2 - \lambda_2 p_2^2) \\
< \;& 0.
\end{aligned}$$

The last inequality is due to the fact that $\lambda_1 p_1^2 < \lambda_2 p_2^2$. For $n = 1$, (28) leads to

$$\int_t^T [\mathcal{G}\pi^{c_1}((u,1) + \lambda_1 p_1]\exp(-\lambda_1(u-t))du < \int_t^T [\mathcal{G}\pi^{c_2}(u,1) + \lambda_1 p_1]\exp(-\lambda_1(u-t))du,$$

and

$$\left[\int_t^T [\mathcal{G}\pi^{c_1}((u,1)+\lambda_1 p_1]\exp(-\lambda_1(u-t))du\right]^+ \leq \left[\int_t^T [\mathcal{G}\pi^{c_2}(u,1)+\lambda_1 p_1]\exp(-\lambda_1(u-t))du\right]^+.$$

It follows from Theorem 2 that

$$\bar{V}^{c_1}(t,1) \leq \bar{V}^{c_2}(t,1). \tag{29}$$

It is apparent that if $\bar{V}^{c_1}(t,1) > 0$, then the above inequality is strict. By the definition of $L(t,n)$, inequalities (28) and (29) lead to

$$L^{c_1}(t,2) \leq L^{c_2}(t,2).$$

Suppose the claim remains true for all $n < k$. That is,

$$\bar{V}^{c_1}(t,n) \leq \bar{V}^{c_2}(t,n), \quad n < k,$$

and

$$L^{c_1}(t,k) \leq L^{c_2}(t,k).$$

Hence,

$$\int_t^T L^{c_1}(u,k)\exp(-\lambda_1(u-t))du \leq \int_t^T L^{c_2}(u,k)\exp(-\lambda_1(u-t))du,$$

and

$$\left[\int_t^T L^{c_1}(u,k)\exp(-\lambda_1(u-t))du\right]^+ \leq \left[\int_t^T L^{c_2}(u,k)\exp(-\lambda_1(u-t)du\right]^+.$$

This gives

$$\bar{V}^{c_1}(t,k) \leq \bar{V}^{c_2}(t,k),$$

and

$$L^{c_1}(t,k+1) \leq L^{c_2}(t,k+1).$$

The above two inequalities become strict if $\bar{V}^{c_1}(t,k) > 0$. The monotonicity (or strict monotonicity) of x_n^c then follows from its definition. □

Proof of Corollary 5

The potential premium rate satisfies

$$\begin{aligned} \mathcal{G}\pi^c(t,n)+\lambda_1 p_1 &\leq \tilde{r}_1^c - \tilde{r}_2^c \\ &\leq (r_1-r_2) - c(\lambda_1 p_1^2 - \lambda_2 p_2^2). \end{aligned}$$

Obviously the premium becomes negative when

$$c < (r_1-r_2)/(\lambda_1 p_1^2 - \lambda_2 p_2^2).$$

As a result,

$$\bar{V}^c(t,n) = 0, \quad 0 \leq t \leq T, \quad 1 \leq n \leq M.$$

□

Other Results in Section 3

Theorem 8 *If $p_1 > p_2$ and $c < (p_1 + p_2)^{-1}$, then both the optimal premium $\bar{V}^c(t,n)$ and time thresholds y_n^c are decreasing functions of c. In particular, if there exist c_1 and c_2 with $c_1 < c_2$ such that $0 < y_n^{c_2}$ and $y_n^{c_1} < T$, then the monotonicity of $\bar{V}^c(t,n)$ and y_n^c is strict on the interval $[c_1, c_2]$.*

Proof. The proof is similar to the proof of Theorem 6 and is omitted. □

Corollary 9 *If $p_1 > p_2$ and $c \leq (r_1 - r_2)/(\lambda_1 p_1^2 - \lambda_2 p_2^2)$, then for all $n \geq 1$ and $0 \leq t \leq T$, $\bar{V}^c(t,n) > 0$ and $y_n^c = 0$; i.e., the higher price will remain unchanged during the entire sales season.*

Proof. From the proof of Corollary 5,

$$\mathcal{G}\pi^c(t,n) + \lambda_1 p_1 \geq (r_1 - r_2) - c(\lambda_1 p_1^2 - \lambda_2 p_2^2).$$

The right side of the inequality becomes positive when

$$c < (r_1 - r_2)/(\lambda_1 p_1^2 - \lambda_2 p_2^2).$$

Consequently,

$$\mathcal{G}\pi^c(t,n) + \lambda_1 p_1 > 0, \qquad 0 \leq t < T,$$

and

$$\bar{V}^c(t,n) > 0, \quad 0 \leq t < T, \quad n = 1, \cdots, M.$$

□