Theory of Optimal Search

Lawrence D. Stone

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Theory of Optimal Search

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Preface

This book deals with the problem of optimal allocation of effort to detect a target. A Bayesian approach is taken in which it is assumed that there is a prior distribution for the target's location which is known to the searcher as well as a function which relates the conditional probability of detecting a target given it is located at a point (or in a cell) to the effort applied there. Problems involving computer search for the maximum of a function do not, for the most part, conform to this framework and are not addressed.

The allocation problems considered are all one-sided in the sense that only the searcher chooses how to allocate effort. For example, the target is allowed to move but not to evade. Thus, pursuit and evasion problems are not considered.

The primary focus of the book is on problems in which the target is stationary. For this case, we use a generalized Lagrange multiplier technique, which allows inequality constraints and does not require differentiability assumptions, to provide a unified method for finding optimal search allocations. This method is also extended to find optimal plans in some cases involving false and moving targets. For the case of Markovian target motion, results are, for the most part, presented without proof, and the reader is referred to the appropriate papers for proofs.

There are a large number of search-related problems, and this book does not claim to cover them all. Instead, the following basic classes of search problems are considered: search for a stationary target, search in the presence of false targets, optimal search and stop, and search for targets with conditionally deterministic motion and Markovian motion. Approximation of optimal plans is also briefly considered. Work on many of the interesting variations of these problems is referenced in the notes at the end of chapters.
Since this book gives the first unified presentation of the basic results in search theory, the notes also attempt to credit these results to the people who originally published them. The search-related problem of modeling the detection process is not considered in this book.

The choice of material included in this book was strongly influenced by experience in major search operations and in constructing computerized search planning systems. For example, the material on uncertain sweep width in Chapter II and that on false targets in Chapter VI was developed in response to problems encountered in the 1968 search for the missing submarine Scorpion. The problems in Chapter VIII on conditionally deterministic motion were first investigated when the problem of searching for a drifting target was considered for a computer-assisted search planning program developed for the U.S. Coast Guard.

The book is written for a person interested in operations research who has a strong mathematics background. The presentation of material in the early chapters is leisurely with proofs and motivation given in detail. In the later chapters the presentation is more condensed. The introductory chapter contains a chapter-by-chapter summary of the book and a guide to the logical dependence of chapters.

Lawrence D. Stone

June 1975
Acknowledgments

This book was written during the academic year 1973–1974 which I was invited to spend at the Naval War College in Newport, Rhode Island, for this purpose. The opportunity to do this was initiated by the suggestion of Warren F. Rogers, Chairman of the Department of Management at the Naval War College. The work in search theory by myself and my colleagues at Daniel H. Wagner, Associates, which provided the knowledge and experience required to write this book, has been primarily supported by the Naval Analysis Programs of the Office of Naval Research under the sponsorship of Robert J. Miller and J. Randolph Simpson. Substantial portions of this work are referenced in or adapted to the book.

I wish to acknowledge the debt that this book owes to the opportunity that my colleagues and I have had to participate in various major search operations. In fact, significant portions of the theory presented in this book have been motivated by and in part applied to these searches. This involvement in major search operations began with the participation of Henry R. Richardson in the 1966 search for the H-bomb lost in the Mediterranean near Palomares, Spain. Further experience in search was obtained under contract to the Navy's Deep Submergence Systems Project, headed by John P. Craven. This experience was deepened by participation in the 1968 search for the missing United States nuclear submarine Scorpion and the other projects discussed below. As a result of the knowledge gained in these operations, we prepared a manual for the operations analysis of search for submerged stationary objects (Richardson et al., 1971) in conjunction with Frank A. Andrews for the Navy's Supervisor of Salvage. At this time, the Office of Naval Research began to support work for the purpose of solving some of the theoretical problems presented by the Scorpion search, notably those involving
uncertain sweep width and false targets. Subsequent work on search problems has included the development of computer programs to assist the U.S. Coast Guard in their search and rescue operations. This work has led to a broader understanding of search problems and to some of the theoretical developments on moving-target problems presented in Chapter VIII. Recently analysis support was provided to the U.S. Navy in order to help them assist the Egyptians in clearing ordnance from the Suez Canal.

I would like to thank Henry R. Richardson for writing a first draft of the section of Chapter II which deals with uncertain sweep width. The book benefited greatly from comments made by the many people who read all or part of the manuscript. Among these are Warren F. Rogers, George F. Brown, Richmond M. Lloyd, and Chantee Lewis of the Naval War College; Daniel H. Wagner, Henry R. Richardson, and Bernard J. McCabe of Wagner Associates; and Joseph P. Kadane of Carnegie-Mellon University.

I greatly appreciate the patient and good natured efforts of Grace P. McCrane, who typed the major portion of the original draft. I also wish to thank Jane James, Linda D. Johnson, Colleen V. Penlington, Lisa L. DiLullo, and Christine M. McAllister for their assistance in preparing the manuscript.

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Author's Preface to the Second Edition

This book was awarded the 1975 Lanchester Prize by the Operations Research Society of America. Since then there have been major developments in search theory and its applications. On the theoretical side, there has been significant progress made in solving the problem of optimal search for a moving target. Appendix C reviews this progress and provides a bibliography of references to work in this area. In the area of optimal search for stationary targets, the results published in the first edition have stood the test of time in the sense that more recent results have tended to be modest generalizations of existing ones, often proved with great effort.

On the applied side, the development of cheap and powerful microcomputers with high resolution color graphics has allowed the development of real-time interactive search planning systems. Because of these computer developments, the trend in search theory is toward algorithms for computers and away from the theorem-proof style of presentation given in this book.

Appendix D contains some corrections to the first edition.

LAWRENCE D. STONE

February 1989
The search process is inherently a nervous one. Either you will find the "target" or you won't. This involves more stress than the continuous penalties or payoffs associated with dullness or brilliance in dealing with problems such as scheduling or logistics. This discontinuity makes search a little like litigation. During an actual "case" there is a sense of urgency and emergency. This stress can trigger a major, sometimes frantic, effort. Experts can be mobilized. Armies (or navies) can be sent scurrying around. A nervous principal or client can make intuitive decisions that are painfully wrong. In short, search theory is a delightful challenge for operations research.

Stone's book, first published in 1975, was and is an elegant response to that challenge. As Stone himself says, B. O. Koopman pioneered the application of coherent mathematical process to the search problem during World War II. Stone's book exploits the advances that we have witnessed in pure mathematics since then. The origins of these advances -- from men like Lebesgue and Borel -- clearly antedated the War; but it was not until the 1940's that the mathematical community began to experience the energetic and pervasive consequences of the pre-war concepts.

Stone wasn't kidding when he stated in his 1975 introduction that the book was written for those with "a strong mathematics background". Even those with first-class graduate credentials in operations research will find themselves
stretched to appreciate clearly the underlying assumptions for his results, and equally stretched to digest the results themselves. From this community, it will be the truly remarkable exception that will comprehend -- much less enjoy -- his proofs and existence theorems. Even at that, the stretching will be good for us, and the results constitute an invaluable definitive set of solutions to central problems in search theory.

Oddly enough, a collateral target of this book should be pure mathematicians themselves. We have already noted that it takes a real mathematician to appreciate and enjoy the subtleties and rigor that the book presents. There is a real dearth of monographs that reach this far into sophisticated mathematics in support of real problems in the world of operations research. In our day, mathematicians had a dismaying tendency to excise any intuitive or applications-oriented origins of what were presented to the mathematical community as elegant, abstract ideas. As a result, many mathematics students were deprived of the very stimulations that fuel some of the best mathematical advances. This book should provide a welcome meal for the reality-starved pure mathematician.

JOHN D. KETTELLE

February 1989
Introduction

We begin by discussing the objectives of this book and its intended audience. A brief description of the setting of search theory is given and an example is presented to illustrate some of the concepts and methods of the theory. Then, the results of the book are outlined in a chapter-by-chapter fashion. Finally, there is a guide to the logical dependence of the chapters.

Search theory is one of the oldest areas of operations research. The initial developments were made by Bernard Koopman and his colleagues in the Anti-Submarine Warfare Operations Research Group of the US Navy during World War II to provide efficient methods of detecting submarines. Search theory is still widely used for this purpose. In addition, the theory has been used by the US Navy to plan searches for objects such as the H-bomb lost in the ocean near Palomares, Spain, in 1966 and the submarine Scorpion lost in 1968, as well as numerous, less well-known objects. The US Coast Guard is also using search theory to plan some of its more complicated search and rescue efforts.

Other possible areas of application of search theory include prospecting and allocating marketing or law enforcement effort (see Larson, 1972). Searches for persons lost on land also fit into the framework of the theory.

OBJECTIVES

In spite of its history and usefulness, no unified and comprehensive presentation of the results in search theory has previously been published. One
objective of this book is to make such a presentation of the major results in the theory of optimal search. Prior to the writing of this book, the results of search theory have appeared scattered throughout the literature of operations research, applied mathematics, statistics, and optimization theory. The methods used in one article appear, on the surface, to be unrelated to the methods in another. In this book, a basic optimization technique, i.e., use of Lagrange multipliers and maximization of Lagrangians, is presented and used to solve most of the problems of optimal allocation encountered here.

Another objective is to give a presentation of search theory appropriate to someone interested in applying the basic theory rather than becoming a specialist. This is done in Chapters I, II, IV, and VII, which contain results that allow the reader to find optimal search plans for the most basic search problem, that of finding a stationary target when no false targets are present.

The scope of the book is not encyclopedic. It contains what the author considers to be the core of the subject as it now stands. In selecting results to present, the author has tried to include those applicable to a wide range of search problems or important to the development of search theory. In addition, there is an emphasis on results that allow one to compute search plans. However, certain results have not been presented because their development would be too technical and tedious for this book. This is particularly true for searches including false or moving targets. Often these results will be described and referenced for the interested reader.

It is important to emphasize that the theory of optimal search is still developing, particularly in the areas involving moving or false targets. It is hoped that this book will outline what is presently known and provide impetus for researchers to extend the theory.

The presentation of the book is designed for an operations researcher with a strong mathematics background. In particular, it is assumed that the reader has had a graduate level course in probability theory and a solid background in real analysis. Knowledge of Lebesgue integration is desirable but not essential. In the early chapters the presentation is leisurely, with proofs and motivation given in detail. In the later chapters the presentation is more condensed. Parts of the book that are more technical than the general level of the chapter in which they appear are indicated by an asterisk. These may be skipped without impairing the reader’s ability to follow most of the subsequent material.

THE PROBLEM OF OPTIMAL SEARCH

The problem of optimal search begins with an object of interest, the target, which the searcher wishes to find. The target is assumed to be located either
at a point in Euclidean $n$-space, $X$, or in one of a possibly infinite collection of cells, $J$. The search space $X$ is called continuous and the space $J$ discrete. While the target's exact position is unknown, it is assumed that there is a probability distribution, known to the searcher, for the target's position at time 0. For most of the search problems considered in this book, the target is assumed to be stationary.

There is a detection function $b$ that relates effort applied in a region or cell to the probability of detecting the target given that it is in that region or cell. In addition, there is a constraint on effort. The basic search problem is to find an allocation of effort in space that maximizes the probability of detecting the target subject to the given constraint on effort.

In some cases optimal search for moving targets will be investigated. However, situations in which the target is evading are not considered. Mathematically speaking, only one-sided optimization problems are considered; that is, the only choice to be made is the allocation of the searcher's effort. If the target moves, it does so independently of the searcher's actions. Thus, the target may move randomly according to distributions assumed known to the searcher, but it is not allowed to take evasive action related to the searcher's action.

In addition to maximizing the probability of detection within a fixed constraint on effort, other measures of optimization are considered. Attention is given to search plans that yield the maximum probability of detection at each time instant during an interval of time or that minimize the mean time to find the target.

In order to illustrate the concepts just discussed in a more specific manner, a simple example of a search problem and its solution are given.

**EXAMPLE**

The following simple example illustrates some of the basic concepts and techniques of search theory. The example also provides a convenient background for the discussion of results given later in this chapter.

Consider an idealized search with two cells, and suppose the target has probability $p(1) = \frac{3}{4}$ of being in cell 1 and probability $p(2) = \frac{1}{4}$ of being in cell 2 (see Fig. 1). One could think of the cells as drawers filled with coins and the searcher as looking for a particular coin, which is very valuable. He feels the valuable coin is more likely to be in drawer 1 than in drawer 2.

Suppose there is a fixed amount of time $K$ in which to find the target. How should time be divided between the two cells in order to maximize the probability of detecting the target (i.e., finding the coin)? To answer this question, it is necessary to assume there is a detection function $b$ that relates time spent
looking in a cell to probability of detecting the target given that it is in the cell. Assume that the detection function is exponential, i.e.,

\[ b(z) = 1 - e^{-z} \quad \text{for } z \geq 0, \]  

where \( z \) is measured in hours. Thus, if \( z \) hours are spent looking in cell 1, there is probability \( b(z) \) of finding the valuable coin given that it is in cell 1.

Without justifying the exact form of the detection function \( b \), we note that it has two intuitively plausible features. First, if the target is in a given cell and some time is spent looking in that cell, then there is always positive probability that the target will be overlooked. In the case of the coins, this could result from the possibility that the searcher may not recognize the valuable coin even if he looks right at it. The second is a saturation effect or law of diminishing rate of returns. This is evidenced by the decreasing nature of the derivative \( b'(z) = e^{-z} \) of the detection function. The result is that the longer one looks in a given cell, the slower the rate of increase of probability of detection.

Suppose that \( z_1 \) time is spent looking in cell 1 and \( z_2 \) time in cell 2. The probability of detecting the target with this allocation of time is

\[ p(1)(1 - e^{-z_1}) + p(2)(1 - e^{-z_2}) \]

and the total time (or cost) required by this allocation is \( z_1 + z_2 \).

Let \( f \) be an allocation, i.e., \( f(j) \) tells the amount of time spent looking in cell \( j \) for \( j = 1, 2 \). Define

\[ P[f] = p(1)b(f(1)) + p(2)b(f(2)), \]  

\[ C[f] = f(1) + f(2). \]  

Then \( P[f] \) gives the probability of detection and \( C[f] \) the cost (or total time) associated with the allocation \( f \). Of course, \( f(j) \geq 0 \) for \( j = 1, 2 \). In mathematical terms, this simple search problem becomes that of finding an allocation \( f^* \) such that \( C[f^*] \leq K \) and

\[ P[f^*] = \text{maximum of } P[f] \text{ over all allocations } f \text{ such that } C[f] \leq K. \]  

Such an \( f^* \) is called optimal for cost \( K \).

Suppose the searcher has spent \( z_1 \) time looking in cell 1 and \( z_2 \) time looking in cell 2 and is considering spending a small increment of time \( h \) looking
Example

in cell 1 again. The increase in probability resulting from this increment is approximately

\[ p(1)b'(z_1)h. \]

If the increment were added to cell 2, the increase would be approximately

\[ p(2)b'(z_2)h. \]

In the short term the searcher would benefit most from placing the increment in the cell having the highest value of \( p(j)b'(z,). \)

Let \( r \) be the probability that the target has not been detected after spending \( z_1 \) time looking in cell 1 and \( z_2 \) looking in cell 2. Since \( b'(z) = e^{-z} = 1 - b(z) \), it follows that

\[ p(j)b'(z)/r = p(j)[1 - b(z,j)]/r \]

is the posterior probability that the target is in cell \( j \) given that the target has not been detected. Thus, for an exponential detection function, searching in the cell with the highest value of \( p(j)b'(z,j) \) is equivalent to searching in the cell with the highest posterior probability. This results from the following property of the exponential detection function:

\[ \Pr(\text{detection in time } z + h \mid \text{failure by time } z) = \frac{b(z + h) - b(z)}{1 - b(z)} = 1 - e^{-h}, \]

i.e., the probability of detecting in the next increment of time \( h \) given failure to detect previously is independent of the amount of time spent searching. In this sense the exponential distribution does not “remember” how much effort has been expended searching in a given cell. As the search progresses and the target is not found, only the posterior target location probabilities change. Thus it is clear that the optimal short-term policy for an exponential detection function should be to place the next small increment of effort in the cell or cells with the highest posterior probabilities.

As is well known, the exponential detection function is the only one having this lack of memory. For other detection functions one would expect that it is necessary to take account of both the posterior target probability and the posterior detection function given failure to detect by time \( z_j \). In fact, the probability of detecting the target with an increment \( h \) of time spent in cell \( j \) given failure to detect the target previously is

\[ \frac{1}{r} p(j)[b(z, + h) - b(z,j)]. \]

As \( h \) becomes small, the above expression is approximately equal to

\[ p(j)b'(z,j)h. \]
Thus, taking into account both the posterior target probabilities and the posterior detection function, one obtains the criterion of searching where \( p(j)b'(z_j) \) is highest.

It still remains to show that the policy that yields the maximum short-term gain also produces an optimal long-term policy. To do this, we define

\[
\rho(j, z) = p(j)b'(z) \quad \text{for } j = 1, 2, \quad z \geq 0.
\]

Then \( \rho \) is called the rate of return function. Consider a search policy that always places the next "small" increment of effort in the cell having the highest rate of return, considering the time previously spent looking in that cell. That is, the next increment goes into the cell \( j^* \) such that

\[
\rho(j^*, z_t) = \max_{j=1,2} \rho(j, z_t),
\]

where \( z_t \) gives the amount of time previously spent looking in cell \( j \). Such a search policy is called locally optimal.

At time 0 when no search has been made in either cell,

\[
\rho(1, 0) = p(1)b'(0) = \frac{3}{2}, \quad \rho(2, 0) = p(2)b'(0) = \frac{1}{2}.
\]

Thus, a locally optimal policy calls for looking solely in cell 1 until time \( s \), such that

\[
\rho(1, s) = \frac{3}{2}e^{-s} = \frac{1}{2} = \rho(2, 0),
\]

i.e.,

\[
s = \ln 2.
\]

In order to continue the locally optimal policy, the additional times \( d_1 \) and \( d_2 \) spent looking in cell 1 and 2, respectively, must be split so that

\[
\rho(1, s + d_1) = \frac{3}{2}e^{-(s + d_1)} = \frac{1}{2}e^{-d_2} = \rho(2, d_2),
\]

i.e.,

\[
d_1 = d_2.
\]

This search policy may be described as follows: Let \( \varphi^* (j, t) \) be the amount of time out of the first \( t \) hours spent looking in cell \( j \) for \( j = 1, 2 \). Then

\[
\varphi^* (1, t) = \begin{cases} 
 t & \text{for } 0 \leq t \leq \ln 2, \\
 \frac{1}{2}(t + \ln 2) & \text{for } \ln 2 < t < \infty,
\end{cases}
\]

\[
\varphi^* (2, t) = \begin{cases} 
 0 & \text{for } 0 \leq t \leq \ln 2, \\
 \frac{1}{2}(t - \ln 2) & \text{for } \ln 2 < t < \infty.
\end{cases}
\]  \hspace{1cm} (5)

Suppose that \( K = 4 \text{ hr} \) is the amount of time available for the search. Let

\[
f^*(1) = \varphi^* (1, 4) = 2 + \frac{1}{2} \ln 2, \quad f^*(2) = \varphi^* (2, 4) = 2 - \frac{1}{2} \ln 2,
\]
Example

so that $f^*$ gives the allocation resulting from following the plan $\phi^*$ for 4 hr. It is claimed that $f^*$ is optimal for cost $K = 4$ hr. To see this, consider the function $\ell$ defined by

$$\ell(j, \lambda, z) = p(j)b(z) - \lambda z \quad \text{for} \quad j = 1, 2, \quad \lambda \geq 0, \quad z \geq 0.$$ 

This function is called the pointwise Lagrangian, and $\lambda$ is called a Lagrange multiplier. Let

$$\lambda = \rho(1, f^*(1)) = \rho(2, f^*(2)) = \frac{\sqrt{2}}{3} e^{-2}.$$ 

Then one may check that, for $j = 1, 2$,

$$\ell(j, \lambda, f^*(j)) = \max \ell(j, \lambda, z) \quad \text{over} \quad z \geq 0 \quad (6)$$

by taking the derivative $\ell'$ of $\ell$ with respect to $z$ and observing that for $j = 1, 2$,

$$\ell'(j, \lambda, z) \geq 0 \quad \text{for} \quad 0 \leq z \leq f^*(j),$$

$$\leq 0 \quad \text{for} \quad f^*(j) < z < \infty.$$ 

The fact that $f^*$ satisfies (6) allows one to show that $f^*$ is optimal for cost $K = 4$ as follows. Suppose that $f$ is an allocation such that $C[f] \leq K = 4$ hours. Then by (6),

$$\ell(j, \lambda, f^*(j)) \geq \ell(j, \lambda, f(j)) \quad \text{for} \quad j = 1, 2.$$ 

(7)

Summing both sides of (7), one obtains

$$P[f^*] - \lambda C[f^*] \geq P[f] - \lambda C[f],$$

which implies

$$P[f^*] - P[f] \geq \lambda (C[f^*] - C[f]) \geq 0,$$

where the last inequality follows from $\lambda \geq 0$ and $K = C[f^*] \geq C[f]$. Thus $f^*$ is optimal for cost $K = 4$ hr. One may check that $P[f^*] \approx 0.87$.

If $\lambda$ and $f^*$ satisfy (6) for $j = 1, 2$, then we say that $(\lambda, f^*)$ maximizes the pointwise Lagrangian. By setting $\lambda = \rho(1, \phi^*(1, t))$ and $f^*(j) = \phi^*(j, t)$ for $j = 1, 2$, one may check that $(\lambda, f^*)$ maximizes the pointwise Lagrangian and that $f^*$ is optimal for cost $K = t$ for $t \geq 0$. Because of this, the plan $\phi^*$ is called uniformly optimal. Let $\mu(\phi)$ be the mean time to find the target using plan $\phi$ and let $\phi(\cdot, t)$ be the allocation of search effort resulting from following plan $\phi$, for time $t$. Since

$$\mu(\phi) = \int_0^\infty (1 - P[\phi(\cdot, t)]) \, dt,$$

and $\phi^*$ is uniformly optimal (i.e., $P[\phi^*(\cdot, t)] \geq P[\phi(\cdot, t)]$ for $t \geq 0$), one can
see that $\phi^*$ minimizes the mean time to find the target among all search plans, such that $C[\phi(\cdot, t)] = t$ for $t \geq 0$.

**OUTLINE OF CHAPTERS**

**Chapter I: Search Model**

The basic search problem is defined and motivated here. Target distributions and methods for generating them are discussed. Detection functions are defined and examples of detection functions are derived for some special situations. The class of regular detection functions is introduced, where a regular detection function $b$ is one such that $b(0) = 0$ and $b'$ is continuous, positive, and strictly decreasing. (Note that the detection function in the preceding example is regular.) In Section 1.3, the basic problem of optimal search is stated in a concise and mathematical form for both the search space $X$, Euclidean $n$-space, and the search space $J$, a possibly infinite subset of the positive integers. Each $j \in J$ may be thought of as representing a cell where the target may be located.

**Chapter II: Uniformly Optimal Search Plans**

Section 2.1 presents optimization techniques, based on Lagrange multipliers, that do not require differentiability assumptions. In Theorems 2.1.2 and 2.1.3, it is shown that maximization of the pointwise Lagrangian is sufficient for constrained optimality. Theorems 2.1.4 and 2.1.5 give conditions under which maximization of a pointwise Lagrangian is necessary and sufficient for constrained optimization. There is an interesting difference here between the continuous and discrete search spaces. For the discrete space, one must assume that the detection function $b$ is concave to guarantee that maximizing a pointwise Lagrangian is necessary for constrained optimality. For a continuous search space, no such assumption is required.

The results of Section 2.1 permit search problems to be attacked by maximizing pointwise Lagrangians. Looking at the preceding example, we see that this problem is equivalent to maximizing a real-valued function of a real variable. The techniques developed in Section 2.1 are basic to the presentation in this book and are used in various forms to solve most of the problems considered. Exceptions to this are the problems of optimal stopping considered in Chapter V and searches for targets with Markovian motion discussed in Chapter IX.

Section 2.2 finds uniformly optimal search plans when the detection function is regular by using the Lagrange multiplier techniques of Section 2.1. The uniformly optimal plan is computed and its properties are studied for the case
of a circular normal target distribution having density function $p$ defined on the plane and an exponential detection function $b$, i.e.,

$$p(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \quad \text{for } x = (x_1, x_2) \in \mathcal{X},$$

and

$$b(z) = 1 - e^{-z} \quad \text{for } z \geq 0.$$

In Example 2.2.8 an algorithm is given for optimally allocating effort to a finite number of cells when the detection function is exponential. Example 2.2.9 shows that if the detection function is not regular, a uniformly optimal plan may not exist for a discrete search space.

In Section 2.3 application is made of the results of Section 2.2 to find uniformly optimal plans when the search sensor has an uncertain detection capability. In Example 2.3.5 the uniformly optimal search plan is calculated for a search in which the target distribution is circular normal and the sweep width of the search sensor is gamma distributed. The mean time to detect the target using the optimal plan is compared to suboptimal plans, which do not take full account of the uncertainty in sweep width. It is shown that substantial penalties in mean time are possible when using these suboptimal plans.

Section 2.4 considers more general situations in which uniformly optimal plans exist. In particular, when the search space is $\mathcal{X}$, it is shown that a uniformly optimal search plan exists whenever the detection function is right continuous and increasing. For discrete search space, uniformly optimal plans are shown to exist whenever the detection function is concave and continuous.

**Chapter III: Properties of Optimal Search Plans**

Properties of optimal search plans are investigated in this chapter. In Section 3.1 an interpretation of Lagrange multipliers is given in terms of marginal rates of return, and it is shown that locally optimal search plans are uniformly optimal when the detection function is regular. The proof is a generalization of the one given in the preceding example. In Section 3.2 it is shown that always searching where the posterior target distribution is the highest produces a uniformly optimal search plan if and only if the detection function is homogeneous (i.e., not space dependent) and exponential. Section 3.3 considers search plans that are obtained in an incremental fashion by allocating a number of increments of effort so that each increment yields the maximum increase in probability of detection considering the application of the previous increments. It is shown that when the search space is $\mathcal{X}$, this
incrementally optimal allocation always produces an optimal allocation of the total effort involved.

In Section 3.4 the question of incremental and total optimality is considered in the context of multiple constraints. In this case, Example 3.4.4 shows that incrementally optimal allocations may fail to be totally optimal even if the search space is $X$ and the detection and cost functions are linear. This is in strong contrast to the case of a single constraint, where the result is true in great generality for the search space $X$.

**Chapter IV: Search with Discrete Effort**

This chapter considers discrete search. In this case effort must be applied in discrete units in any cell. These units are called *looks*. Let

$$\beta(j, k) = \text{probability of detecting the target on but not before the } k\text{th look in cell } j \text{ given that the target is in cell } j,$$

$$\gamma(j, k) = \text{cost of } k\text{th look in cell } j.$$

A search plan that always takes the next look in a cell $j^*$ such that

$$\frac{p(j^*)\beta(j^*, k_r + 1)}{\gamma(j^*, k_r + 1)} = \max_{j \in J} \frac{p(j)\beta(j, k) + 1}{\gamma(j, k) + 1},$$

where $k_r$ is the number of looks previously made in cell $j$, is called locally optimal. The ratio $p(j)\beta(j, k)/\gamma(j, k)$ is the discrete counterpart of $p(j, z)$. Theorem 4.2.5 shows that if $\gamma$ is constant and if $p(j)\beta(j, k)$ is decreasing, then a locally optimal search plan is uniformly optimal. Observe that requiring $p(j)\beta(j, k)$ to be decreasing is the discrete analog of requiring the detection function $b$ to be regular. Theorem 4.3.2 shows that if $\gamma$ is bounded away from 0 and if $p(j)\beta(j, k)/\gamma(j, k)$ is decreasing, then a locally optimal plan minimizes the mean cost to find the target.

Section 4.4 considers whereabouts searches, in which the objective is to say which cell contains the target within a given constraint on cost. There are two ways the searcher may correctly state the target's position: He may detect the target within the cost constraint or, failing to do this, he can correctly guess the cell containing the target. Thus, a whereabouts search strategy is the combination of a detection search strategy and a cell to guess if the search fails to detect the target. In this section it is shown that an optimal whereabouts search proceeds by choosing a cell to guess and performing an optimal detection search in the remaining cells. In special cases, rules are given for choosing the cell to be guessed. In particular, if for some $\alpha > 0$,

$$\beta(j, k) = \alpha(1 - \alpha)^{k-1} \quad \text{for } j \in J, \ k = 1, 2, \ldots,$$

$$\gamma(j, k) = 1$$
then one picks a cell \( j^* \) such that \( p(j^*) = \max_{j \in J} p(j) \) to guess in case the detection search fails and performs a locally optimal search in the remaining cells.

**Chapter V: Optimal Search and Stop**

Here the problem of how to search and when to stop in order to maximize expected return is considered when there are a reward for finding the target and costs associated with searching. The combination of a search plan and a stopping time \( s \) determines a search and stop plan as follows. The search plan is followed until either the target is detected or time \( s \) is reached. When either of these events occurs, the search stops. A search and stop plan that maximizes expected return is called optimal.

Section 5.1 considers searches with discrete effort. In addition to satisfying the assumptions of the model of Chapter IV, there is a function \( V \) such that \( V(j) \) gives the value of the target or reward obtained if the target is found in cell \( j \) for \( j \in J \). If \( J \) is finite and \( \gamma \) is bounded away from 0, then Theorem 5.1.1 shows that an optimal search and stop plan exists and the optimal return function satisfies the optimality equation of dynamic programming.

Two special situations are considered first, namely, those in which \( s = 0 \) is optimal and those in which an optimal search and stop plan can be found in the class \( \Xi_\omega \) of plans with \( s = \infty \).

If \( \sum_{j \in J} p(j)V(j) < \infty \), then Theorem 5.1.3 shows that any plan that minimizes the expected cost to find the target is optimal within \( \Xi_\omega \). Theorem 5.1.6 gives conditions under which an optimal search and stop plan can be found within \( \Xi_\omega \). An immediate consequence of these theorems and Theorem 4.3.2 is that if the conditions of Theorem 5.1.6 are satisfied, the ratios \( p(j)\beta(j, \cdot)/\gamma(j, \cdot) \) are decreasing for \( j \in J \), and \( \gamma \) is bounded away from 0, then any locally optimal search plan coupled with \( s = \infty \) produces an optimal search and stop plan.

Theorem 5.1.8 shows that if \( J \) is finite, \( \gamma \) is bounded away from 0 and

\[
\frac{\beta(j, k)}{1 - b(j, k - 1)} V(j) < \gamma(j, k) \quad \text{for} \quad k = 1, 2, \ldots, j \in J,
\]

then \( s = 0 \) is an optimal search and stop plan.

The main result of Section 5.1 is given in Theorem 5.1.11, which shows that if \( V(j) = V_0 \) for \( j \in J \), \( J \) is finite, and \( p(j)\beta(j, \cdot)/\gamma(j, \cdot) \) is decreasing for \( j \in J \), then an optimal search and stop plan follows a locally optimal plan as long as it continues to search. Lemma 5.1.9 implies that all locally optimal plans with the same stopping time have the same expected return. Thus, under the conditions of Theorem 5.1.11, the problem of optimal search and stop is reduced to the problem of optimal stopping when using a locally optimal search plan. Unfortunately, the optimal stopping problem is difficult to solve.
Section 5.2 considers optimal search and stop when effort is continuous. In contrast to Section 5.1, the cost of search is a function of time only and not of the search plan. For a given cumulative effort function $M$, search plans $\varphi$ are restricted to be in the class $\Phi(M)$, i.e.,

$$\int_x \varphi(x, t) \, dx = M(t) \quad \text{for} \quad t \geq 0$$

or $\sum_{j \in J} \varphi(j, t) = M(t)$ for $t \geq 0$ if the search space is $J$. For a fixed stopping time, Theorem 5.2.3 shows that it is optimal to follow a plan that is uniformly optimal within $\Phi(M)$. This reduces the optimal search and stop problem to an optimal stopping problem. Theorem 5.2.5 provides sufficient conditions for a stopping time to be optimal.

**Chapter VI: Search in the Presence of False Targets**

This chapter introduces the problem of search in the presence of false targets. False targets are assumed to be real stationary objects that may be mistaken for the target by the primary or detection sensor. Thus, when a detection is made, an investigation is required to determine whether the contact is a false or real target. Models are developed for the location and detection of false targets, and the distinction between adaptive and nonadaptive plans is discussed.

The optimization criterion used for searches in the presence of false targets is minimization of mean time. For searches in which contact investigation is immediate and conclusive (i.e., contacts must be investigated until identified), an optimal nonadaptive plan is found. It is then shown that immediate contact identification is still optimal in a class of plans in which one is allowed to delay contact investigation.

A class of semiadaptive plans is introduced that takes advantage of the optimal nonadaptive plan and feedback obtained from the search to produce a smaller mean time to find the target than the optimal nonadaptive plan. The semiadaptive plan does not do as well as the optimal adaptive plan, but it is much easier to obtain. In fact, optimal adaptive plans have been found only in very special cases. Applications of semiadaptive plans to searching for multiple targets are also discussed.

**Chapter VII: Approximation of Optimal Plans**

The optimal search plans found in Chapters II and VI may be difficult to calculate without the use of a computer. Even when the optimal plan has a simple analytic form such as the one in Example 2.2.7, it may call for applying very small amounts of effort to some areas. Both of these properties can make the optimal plan difficult to realize in practice.
In order to overcome the second difficulty, Section 7.1 explores approximating plans that are optimal in a restricted class. In this restricted class, effort density must be applied in multiples of a fixed number. Plans that are optimal in this restricted class are appropriate for approximating the plans of Chapter II, which do not involve false targets. The calculation of these approximating plans may still require the use of a computer.

Section 7.2 introduces a class of incremental plans called $\Delta$-plans, which can be used when the target distribution is discrete. For a $\Delta$-plan, one fixes an increment of effort $\Delta$ and proceeds so that at each step the increment $\Delta$ is applied to a single cell. The choice of the cell in which to place the increment is made by examining $\rho$, the marginal rate of return in each of the cells, and choosing the cell having the highest rate. The computation of these rates is easy even for searches involving false targets. Thus, $\Delta$-plans overcome both the difficulties mentioned above. In Section 7.2 it is shown that a $\Delta$-plan approximates the optimal plan in mean time to find the target, in the sense that the mean time resulting from a $\Delta$-plan approaches the mean time of the optimal plan as $\Delta$ approaches 0.

Chapter VIII: Conditionally Deterministic Target Motion

A simple but interesting type of moving-target search problem is considered in this chapter. The target's motion takes place in Euclidean $n$-space and is determined by an $n$-dimensional stochastic parameter $\xi$ such as the target's position at time 0. There is a target motion function $\eta$ such that if $\xi = x$, then $\eta(x, t)$ gives the target's position at time $t$ for $t \geq 0$. Thus, the target motion is deterministic when conditioned on the value of $\xi$. The distribution of $\xi$ is assumed to be known to the searcher and is given by the probability density function $p$.

Let $\eta$ denote the transformation that maps $x$ into $\eta(x, t)$. Then it is assumed that $\eta$, is one-to-one for all $t \geq 0$ and that the Jacobian $J(t, x)$ of $\eta$, evaluated at $x$ is positive for all $x \in X$ and $t \geq 0$. A moving-target search plan is a function $\psi$ such that $\psi(x, t)$ gives the rate at which effort density accumulates at point $x$ at time $t$. There is a detection function $b$ such that if $\xi = x$ and the search plan $\psi$ is followed, then $b \left( \int_0^t \psi(\eta, x, s) ds \right)$ is the probability of detecting the target by time $t$.

Suppose the rate at which search effort may be applied at time $t$ is bounded by $m_2(t)$ for $t \geq 0$. If the Jacobian $J$ factors into space- and time-dependent parts [i.e., $J(x, t) = j(x)u(t)$ for $x \in X$ and $t \geq 0$], then Section 8.2 presents a method of finding uniformly optimal moving-target search plans among the plans $\psi$ such that

$$\int_X \psi(x, t) \, dx \leq m_2(t) \quad \text{for} \quad t \geq 0.$$
Section 8.3 gives sufficient conditions for finding plans that maximize probability of detection among plans $\psi$ such that $\psi(x, t) \leq m_2(t)$ for $x \in X$ and $t \geq 0$ and

$$\int_0^\infty \int_X \psi(x, t) \, dx \, dt \leq K.$$ 

No factorability assumption is made in Section 8.3.

Theorem 8.4.1 gives necessary and sufficient conditions for a plan $\psi^*$ to maximize probability of detection by time $t$ among plans $\psi$ that satisfy

$$\int_X \psi(x, s) \, dx \leq m_2(s) \quad \text{for} \quad s \geq 0.$$ 

The conditions have the form that there exists a nonnegative function $\lambda$ such that

$$\rho(x) b' \left( \int_0^t \psi^*(\eta_u(x), u) \, du \right) = \lambda(s) J(x, s) \quad \text{for} \quad \psi^*(\eta_u(x), s) > 0,$$

$$\leq \lambda(s) J(x, s) \quad \text{otherwise.}$$

In the case of sufficiency, $b$ is required to be concave. Again, no factorability assumption is made. The function $\lambda$ is a generalization of the Lagrange multiplier introduced in Section 2.1. In addition, the above conditions are generalizations of the necessary conditions given in Corollary 2.1.7.

Chapter IX: Markovian Target Motion

Search for targets having Markovian motion is investigated in this chapter. Section 9.1 deals with the simplest type of Markovian motion, a two-cell, discrete-time Markov process. The problem of maximizing the probability of detecting the target in $n$ looks is shown to be a standard dynamic programming problem that is solvable, in principle, for all cases. The problem of minimizing the expected number of looks to find the target is not so straightforward even though the optimal plan satisfies the usual dynamic programming equation. For this case optimal plans are found when detection occurs with probability one if the searcher looks in the cell containing the target. The optimal plan is characterized by a threshold probability $\pi^*$, which depends on the transition matrix of the Markov process. The plan proceeds by looking in cell 1 each time the posterior probability in that cell is greater than $\pi^*$ and looking in cell 2 otherwise until the target is detected.

Let $a_{ij}$ be the probability that the target transitions to cell $j$ at time $n + 1$ given that it is in cell $i$ at time $n$ for $n = 0, 1, \ldots$. If $a_{11} = a_{21}$, then from the first look on, the posterior target location probabilities given failure to detect the target are $\tilde{p}(1) = a_{21}$, $\tilde{p}(2) = 1 - a_{21}$, regardless of the sequence of looks.
or the prior target distribution. This is called the no-learning case. For this case, the plan that minimizes mean time to detect the target is also found.

In Section 9.2 the continuous-time version of the two-cell problem is presented. The nature of the plan that maximizes the probability of detection by time $t$ and a method for finding the plan are discussed.

In Section 9.3 necessary conditions are given for a search plan $\psi^*$ to maximize the probability of detection by time $t$ when the target's position is specified by a continuous-time Markov process $\{X_t, t \geq 0\}$ in Euclidean $n$-space. Let

$$G(x, s, t, \psi) = \Pr\{\text{target not detected during } [s, t] \text{ using plan } \psi \mid X_s = x\}. $$

Let $g(x, s, y, t, \psi)$ be the transition density for the process representing undetected targets, that is,

$$\int_s^t g(x, s, y, t, \psi) \, dy = \Pr\{X_t \in S \text{ and target not detected in } [s, t] \text{ using plan } \psi \mid X_s = x\}. $$

Let $p$ be the probability density of the target's location at time $t = 0$ and let $m_2(s)$ constrain the rate at which search effort may be applied at time $s$ for $s \geq 0$. If $\psi^*$ maximizes the probability of detecting the target by time $T$ among plans $\psi$ that satisfy

$$\int_x \psi(x, s) \, dx \leq m_2(s) \quad \text{for } s \geq 0,$$

then Theorem 9.3.1 gives the following necessary condition, which $\psi^*$ must satisfy: There is a nonnegative function $\lambda$ such that for $0 \leq t \leq T$

$$\int_x g(x, 0, y, t, \psi^*)G(y, t, T, \psi^*)p(x) \, dx = \lambda(t) \quad \text{for } \psi^*(y, t) > 0,$$

$$\leq \lambda(t) \quad \text{for } \psi^*(y, t) = 0.$$

**Appendix A: Reference Theorems**

This appendix gives the statements of some basic theorems from real analysis, which are quoted in the book.

**Appendix B: Necessary Conditions for Constrained Optimization of Separable Functionals**

This appendix proves that maximizing the pointwise Lagrangian is a necessary condition for a constrained optimum involving a separable nonnegative payoff functional and a vector-valued constraint functional whose coordinate functionals are nonnegative and separable. Section B.1 presents
Fig. 2. Guide to dependence of chapters and sections. (*Note:* Chapters are indicated by Roman numerals and sections by Arabic numbers.)

the proof for the case of a discrete search space and Section B.2 for the case of a continuous search space.

**Logical Dependence of Chapters**

Figure 2 gives a guide to the logical dependence of chapters and sections. Thus, if one wishes to read Section 5.1, Optimal Search and Stop with Discrete Effort, the guide shows that one may read Sections 1.3, 2.1, and 4.1–4.3 and then Section 5.1. If one wishes to read Chapter VIII, then the guide shows that one can do this by reading Sections 1.3, 2.1, and 2.2 and proceeding directly to Chapter VIII.